# A METHOD TO DEFINE A MINIMUM-PHASE TRANSFER FUNCTION WITHIN THE BOUNDED REGION OF PHASE-GAIN SPECIFICATIONS

RESEARCH & ANALYSIS SECTION TECH MEMO Nº 55

Prepared Under Contract NAS 8-11111

*for*

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George C. Marshall Space Flight Center
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# 1. ABSTRACT

This report presents a method of defining a minimum-phase transfer function within the bounded regions of phase and gain specifications at several discrete frequencies. The application of complex curve fitting techniques and the principles of gain and phase stability for a transfer function are employed. The developed method is applied to two conditions and six cases; the results show that the method is simple and effective.

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## 3. INTRODUCTION

The material contained in this report was prepared by the Research & Analysis Section of Northrop Space Laboratories in compliance with requirements specified by the Dynamic Stability Section of the Marshall Space Flight Center. In general, the requirement is to develop a method to obtain a realistic stability network configuration from given non-unique phase-gain specification at several discrete frequencies. A systematic design approach to obtain the above objective was defined and reported in the NSL/Huntsville Aero/Astro Tech. Memo. No. 16, May 19, 1964 (Ref. 1). This report contains a portion of the work noted above; specifically, a method is determined for defining a minimum-phase transfer function when bounded regions at various frequency modes of phase and gain are specified. Complex curve fitting techniques and conditions for the stability of transfer functions are incorporated in this report. Figure 3.1 contains a flow chart which depicts the procedure for obtaining stable transfer functions.

Two conditions and six cases has been worked out successfully. It is felt that, if the rules of stability are followed closely (to this effect, a set of templates is used) a minimum-phase transfer function can always be defined within the bounded regions of phase and gain specifications.

After obtaining the most simple minimum-phase transfer function, the next step is to develop a precise method to design a stability network configuration in accordance with the obtained transfer function. It is believed that, with additional study, the network synthesis technique

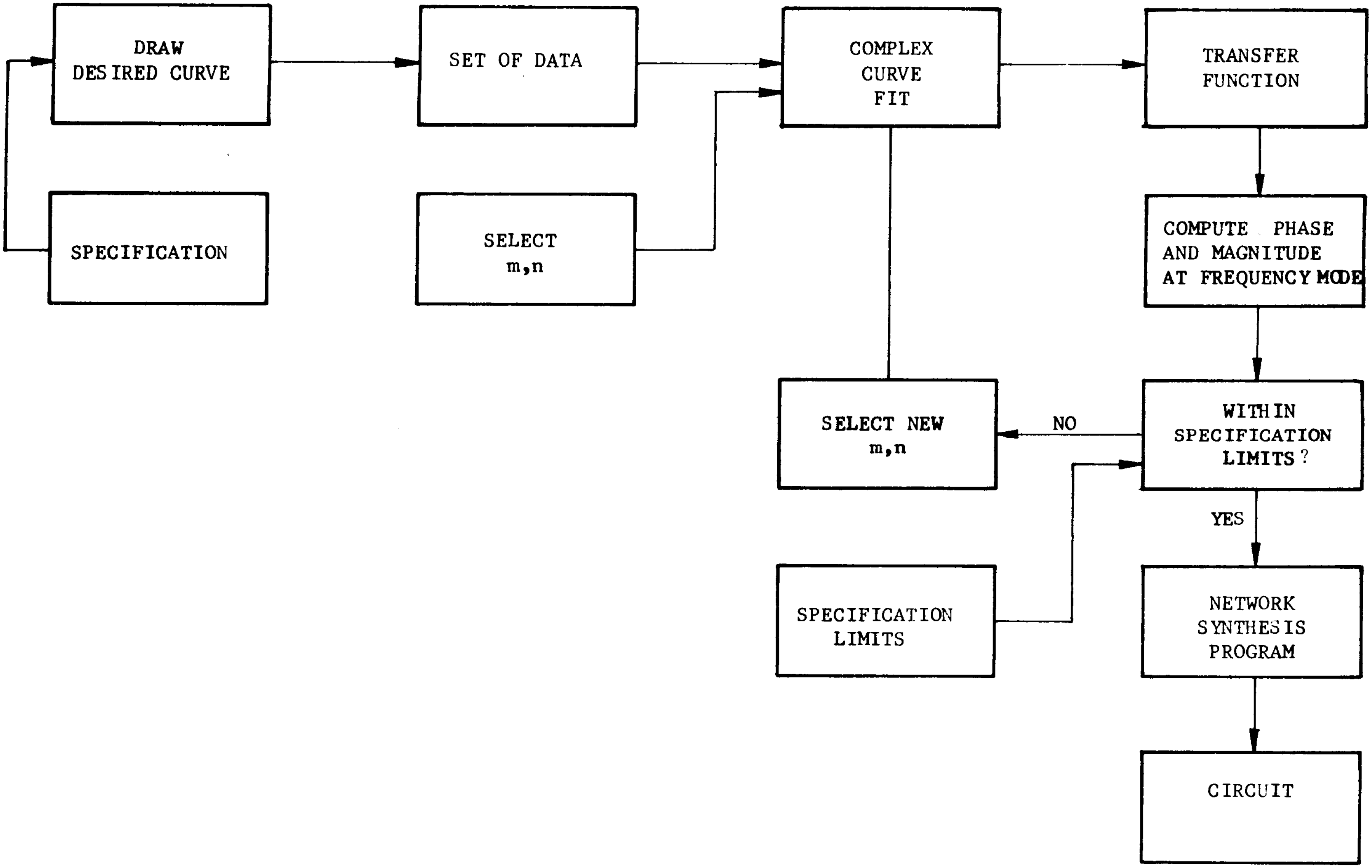



FIGURE 3-1. FLOW CHART WHICH DEPICTS PROCEDURE FOR OBTAINING STABLE TRANSFER FUNCTION

presented in Reference 1 can be refined and fully automated in conjunction with the methods presented in this report.

## 4. MINIMUM-PHASE REQUIREMENTS AND THE RULES OF STABILITY

A system that has no open-loop poles or zeros in the right-half "s" plane is defined as a minimum-phase system. For this type of system all factors of the transfer function are of the form $(1 + TS)$ and/or $(1 + AS + BS^2)$. A system that has open-loop poles or zeros in the right-half "s" plane is defined as a non-minimum-phase system. Thus, for this situation, one or more terms in the transfer function have the form $(1 - TS)$ and/or $(1 \pm AS \mp BS^2)$. Most practical systems are in the minimum-phase category. The methods developed in this report are mainly for minimum-phase systems.

The stability and the corresponding response of a system can be determined from the location of the poles of the transform F(s) in the s plane. The possible positions of the poles are shown in Figure 4-1.

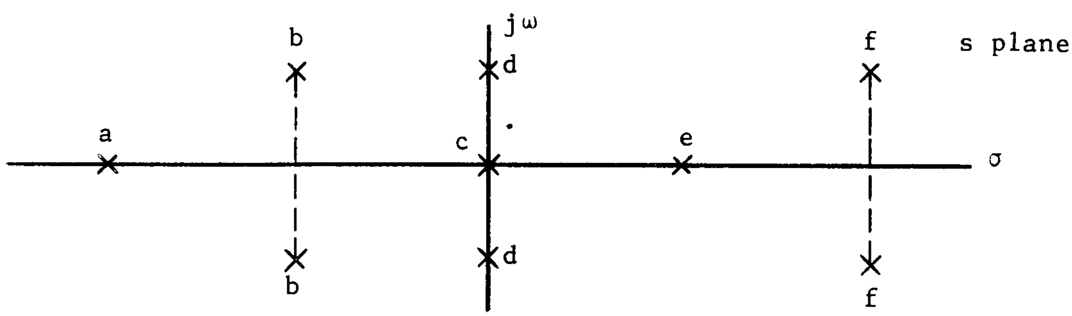



Figure 4-1. Location of poles in the s Plane

For the locations of the poles shown in Fig. 4-1 the responses are given in Table 4-1. These poles are the roots of the denominator of the transform F(s).

Table 4-1. Relation of Response to Location of Poles

| Position of Pole | Form of Response | Characteristics |
|---|---|---|
| a | $K\,e^{-\sigma t}$ | damped exponential |
| b | $K\,e^{-\sigma t}\sin(w_d t + \phi)$ | exponentially damped sinusoid |
| c | $K$ | constant |
| d | $K \sin\,(w_d t + \phi)$ | constant sinusoid |
| e | $K\,e^{+\sigma t}$ | increasing exponential (unstable) |
| f | $K\,e^{+\sigma t}\sin(w_d t + \phi)$ | exponentially increasing sinusoid (unstable) |

where $\sigma$ = real axis, $jw$ = imaginary axis, (frequency)

$K$ = constant, $t$ = time in second

$w_d$ = the damped natural frequency of the oscillation of the transient response.

Poles of the response transform at the origin or on the imaginary axis that are not contributed by the forcing function result in a continuous output. These outputs are undesirable in a control system. Poles in the right-half "s" plane result in transient response that increases with time. This represents an unstable system; therefore, poles in the right-half "s" plane are undesirable.

It has been shown (Ref. 6) that stability of response of $F(s) = \frac{A(s)}{B(s)}$ requires that all roots of B(s) have negative real parts, and for a minimum-phase system, all roots of A(s) have negative real parts also. Since it is useless to find the exact solution when the response is unstable, a simple procedure to determine the existence of roots with positive real parts is desired. Routh's stability criterion and Nyquist's stability criterion are good simple methods of determining the number of roots with positive real parts without actually solving for the roots of B(s) or A(s). However, for the complex curve-fitting technique (Ref. 1) a computer program has been written which computes all zeros and poles of each transfer function automatically. This program will be discussed further in Section 5.

The use of log magnitude and phase angle diagrams are of great value in situations where the transfer function of a system is not known; consequently, the Bode plot of log magnitude and phase is used to define a transfer function.

The stability and approximate degree of stability can be determined from the log magnitude and phase diagram. The stability characteristic is specified in terms of the following quantities:

a. Gain Crossover:

This is the point on the plot of the transfer function at which the magnitude is unity ($\log_{10} F(j\omega) = 0$ db). The frequency at gain crossover is called the phase-margin frequency $\omega_\phi$.

b. Phase Margin:

This is 180° plus the negative trigonometrically considered angle of the transfer function at the gain-crossover point. It is designated as the angle $\gamma$ which can be expressed as

$$\gamma = 180° + \phi$$

c. Phase Crossover:

This is the point on the plot of the transfer function at which the phase angle is 180°. The frequency at which phase crossover occurs is called the gain-margin frequency $\omega_c$.

d. Gain Margin:

This is the additional gain $\alpha$ that just makes the system unstable. Expressed in terms of transfer function at frequency $\omega_c$, it is

$$|F(j\omega_c)| \cdot \alpha = 1$$

In terms of the log magnitudes, in decibels, this is

$$20 \log_{10} \alpha = - 20 \log_{10} |F(j\omega_c)|$$

Figure 4-2 shows gain margin and phase margin and the conditions of stability.

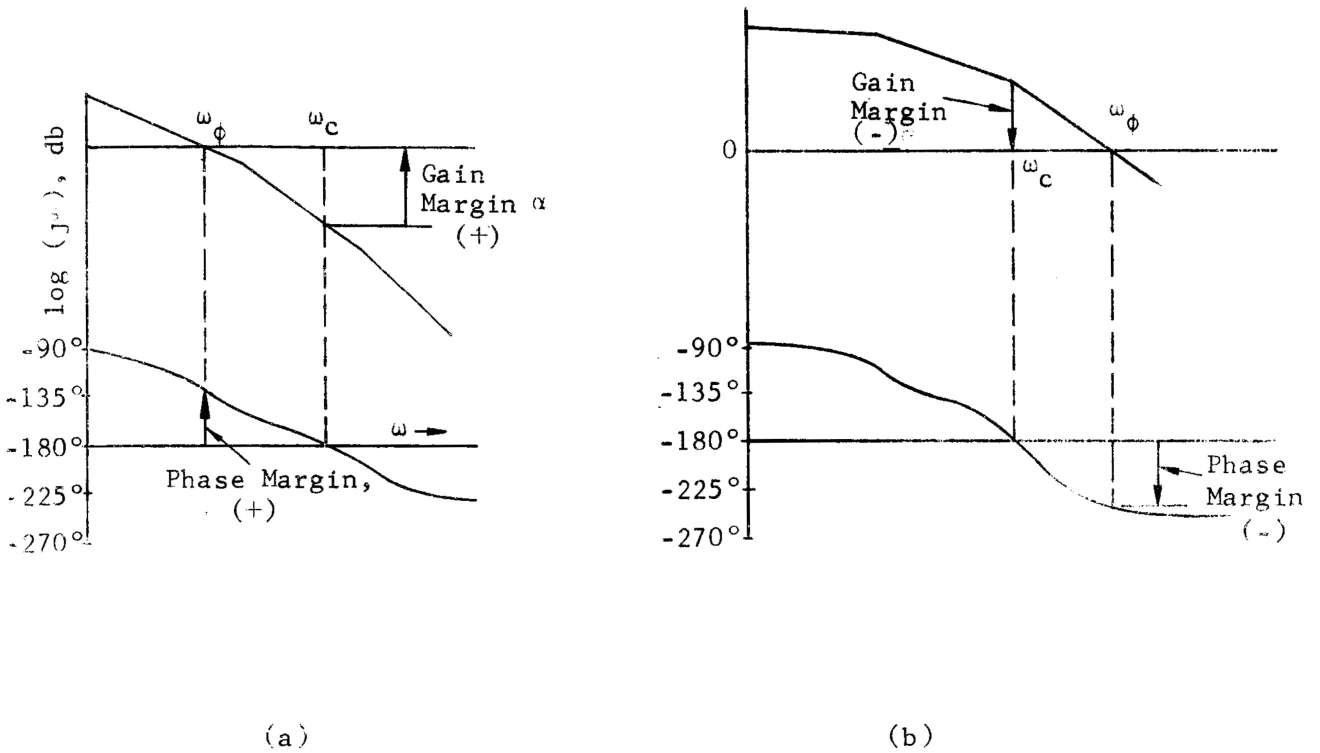



Figure 4-2. Log Magnitude and Phase Diagram, Showing Gain Margin and Phase Margin. ((a) stable, (b) unstable)

The phase margin is the amount of phase shift at frequency $\omega_\phi$ that would just produce instability. For minimum-phase network the phase margin must be positive for a stable system. Satisfactory response is usually obtained with a phase margin of 45 to 60°.

The gain margin is the factor by which the gain must be changed in order to produce instability. The gain margin must be positive when expressed in decibels (greater than unity as a numeric) for a stable system.

The total phase angle of a transfer function at any frequency is closely related to the slope of the log magnitude curve at that frequency. The effectiveness of this relationship for first and second order transfer functions can be realized for frequencies which are normally greater than the phase-margin frequency $\omega_\phi$. A slope of -20 db/decade is related to an angle of -90°; a slope of -40 db/decade is related to an angle of -180°; a slope of -60 db/decade is related to an angle of -270°; etc. Changes of slope at higher and lower frequencies around the particular frequency being considered, contribute to the angle at that frequency. The farther away the changes of slope are from the frequency in question, the less they contribute to the total angle at that frequency.

By placing a line tangent to the log-magnitude curve, it is possible to estimate the approximate value of the phase angle.

As has been seen, the stability of a system required that the phase margin be positive for a minimum-phase system. In order for this to be true, the angle at the gain crossover ($20 \log_{10}|F(j\omega)| = 0$ db) must be greater that -180°. This places a limit on the slope of the log magnitude curve at the gain crossover. The maximum slope of the log magnitude curve at gain crossover should be more positive than -40 db/decade; however, it is common synthesis procedure to assure stability by requiring the log-magnitude diagram to have -20 db/decade of frequency for several octaves of frequency about the gain crossover point. Frequently, the slope at 0-db point is taken as an alternate statement of stability, but it is not always correct.

Bode's attenuation-phase theorem states that the mathematical relationships between the log-magnitude and the phase angle diagram for a minimum-phase system is

$$\theta(\omega_x) = \frac{\pi}{2}\left|\frac{dA}{dN}\right|_o + \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\left|\frac{dA}{dN}\right| - \left|\frac{dA}{dN}\right|_o\right) \ln \coth \left|\frac{v}{2}\right| dv \qquad (4.1)$$

where

$\theta(\omega_x)$ = phase of system at any frequency $\omega_x$,(radian)

$A$ = gain (neper = $\ln|F(j\omega)|$)

$F(j\omega)$ = transfer function of the system

$v = \ln\left(\frac{\omega}{\omega_x}\right)$

$\frac{dA}{dN}$ = slope of log-magnitude diagram (1 neper/unit = 20 db/decade)

$$\int_0^{\omega_o} \frac{A d\omega}{\sqrt{\omega_o^2 - \omega^2(\omega^2 - \omega_x^2)}} + \int_{\omega_o}^{\infty} \frac{\theta d\omega}{\sqrt{\omega^2 - \omega_o^2(\omega^2 - \omega_x^2)}} \qquad (4.2)$$

$$= \frac{\pi}{2} \frac{\theta(\omega_x)}{\omega_x\sqrt{\omega_o^2 - \omega_x^2}} \qquad \omega_x < \omega_o$$

or

$$= -\frac{\pi}{2} \frac{A(\omega_x)}{\omega_x\sqrt{\omega_x^2 - \omega_o^2}} \qquad \omega_x > \omega_o$$

where $\omega_o$ is the frequency above which the phase is known and below which gain characteristic is known.

Equation (4.1) determines the phase characteristic if the gain characteristic over the entire frequency range is known. Equation (4.2) determines the phase and gain characteristic over the unknown portion of the frequency range from the known phase angle over part of the frequency range and the unknown gain over the remaining part of the frequency range. Although these theorems are seldom used for actual calculations because they are more difficult than the methods presented in our report, they do show the interdependence of gain and phase angle for a minimum phase system.

The gain can be adjusted (this raises or lowers the log magnitude curve) to produce a phase margin in the desirable range of 45 - 60°. The phase margin frequency $\omega_\phi$ gives a qualitative indication of the speed of response of a system.

It can be shown that the phase and gain margin is related to the effective damping ratio $\zeta$ of the system. The precise functional relationship between magnitude and phase angle at a given frequency is determined by the break frequency $\omega_n$ and effective damping ratio $\zeta$ when the form of a transfer function is known. Figure 4-3a and 4-3b illustrate this point.

A set of Bode templates have been made in accordance with different values of effective damping ratio $\zeta$. Using these templates, one can draw many phase and gain curves which produce minimum-phase transfer functions. The application of these templates will be discussed further in section 5.3.1.

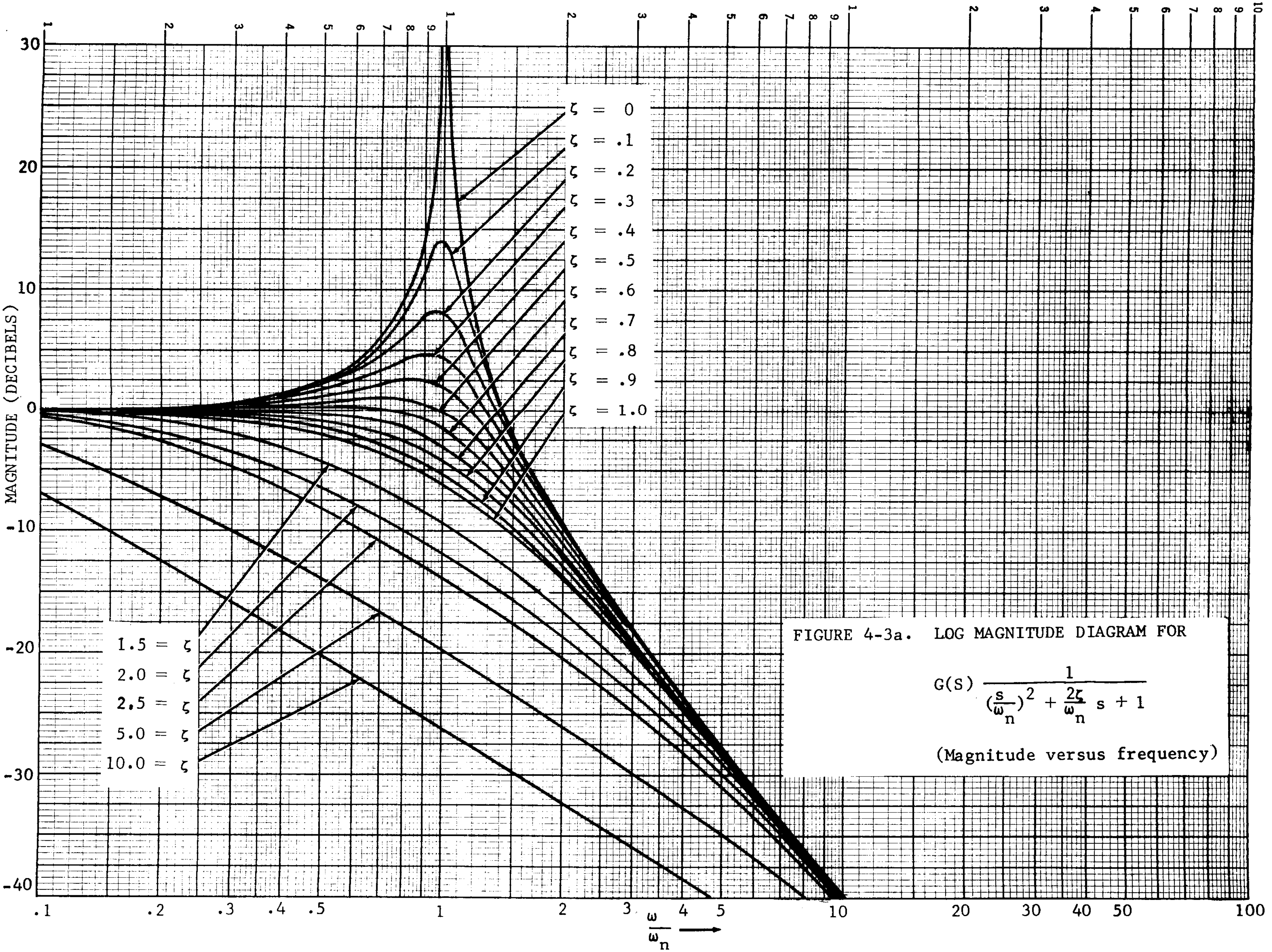



FIGURE 4-3a. LOG MAGNITUDE DIAGRAM FOR

$$G(S) \quad \frac{1}{(\frac{s}{\omega_n})^2 + \frac{2\zeta}{\omega_n} s + 1}$$

(Magnitude versus frequency)

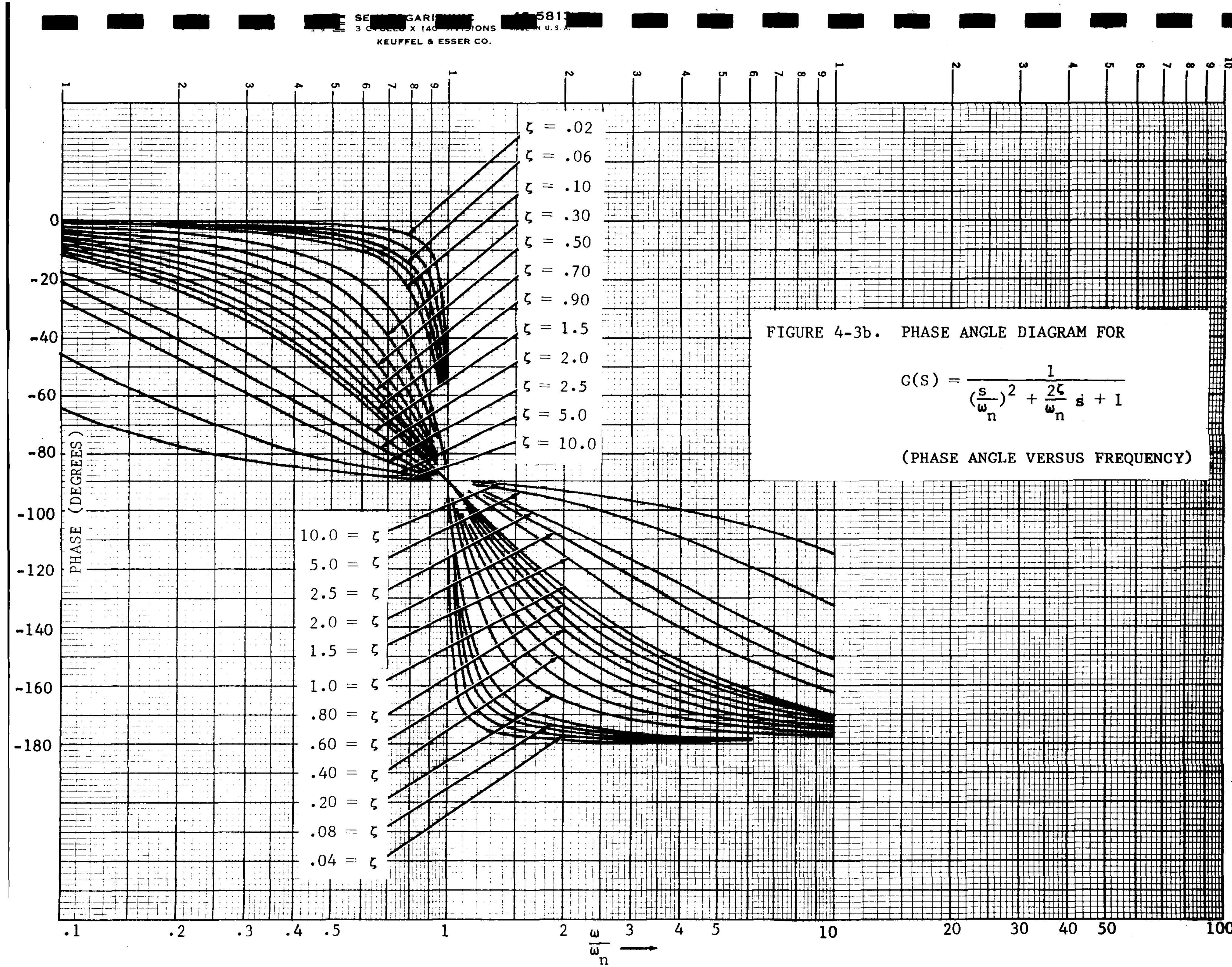



FIGURE 4-3b. PHASE ANGLE DIAGRAM FOR

$$G(S) = \frac{1}{\left(\frac{s}{\omega_n}\right)^2 + \frac{2\zeta}{\omega_n} s + 1}$$

(PHASE ANGLE VERSUS FREQUENCY)

# 5. THE APPLICATION OF COMPLEX CURVE-FITTING TECHNIQUE

## 5.1 Mathematical Model and Computer Programming

The theory and the mathematical model of complex curve fitting technique has been discussed in detail in the NSL/Huntsville AERO/ASTRO Tech. Memo. No. 16, 19 May 1964. In this report it is summarized for the purpose of reference.

The transfer function of a network or a dynamic system is necessary for describing the frequency response characteristics of that network or system, and in general, the response characteristics of a network or a dynamic system at a certain frequency is the ratio of output to input at that frequency.

$$G(j\omega) = \frac{A(j\omega)}{B(j\omega)} \tag{5.1}$$

where $A(j\omega)$ and $B(j\omega)$ can be expressed in terms of complex polynomials. That is

$$G(j\omega) = \frac{p_o + p_1(j\omega) + p_2(j\omega)^2 + p_3(j\omega)^3 + \dots}{q_o + q_1(j\omega) + q_2(j\omega)^2 + q_3(j\omega)^3 + \dots} \tag{5.2a}$$

or

$$G(S) = \frac{p_o + p_1S + p_2S^2 + p_3S^3 + \dots}{q_o + q_1S + q_2S^2 + q_3S^3 + \dots} \tag{5.2b}$$

where

p's and q's are polynomial coefficients

$j = \sqrt{-1}$

$\omega$ = frequency in radian per second

After the separation of real and imaginary parts, equations (5.2a) can be written as

$$G(j\omega) = \frac{\alpha + j\omega\beta}{\sigma + j\omega\tau} \tag{5.3}$$

where

$$\alpha = p_o - p_2\omega^2 + p_4\omega^4 - \ldots,$$
$$\sigma = q_o - q_2\omega^2 + q_4\omega^4 - \ldots,$$
$$\beta = p_1 - p_3\omega^2 + p_5\omega^4 - \ldots,$$
$$\tau = q_1 - q_3\omega^2 + q_5\omega^4 - \ldots,$$

Suppose that the function $F(j\omega)$ is used to designate an "ideal" function; i.e., a function which represents the data exactly. Therefore, $F(j\omega)$ will have real and imaginary components which would coincide exactly with the values indicated by the experimental curve. The function will be

$$F(j\omega) = Re(\omega) + j\ Im(\omega) = R + jI \tag{5.4}$$

The numerical differences between the two functions $G(j\omega)$ and $F(j\omega)$ represents the error in curve-fitting. This error is

$$\varepsilon(j\omega) = F(j\omega) - G(j\omega) \tag{5.5a}$$

$$\varepsilon(j\omega) = F(j\omega) - \frac{(\alpha + j\omega\beta)}{(\sigma + j\omega\tau)} \tag{5.5b}$$

multiplying both sides of equation (5.5b) by $(\sigma + j\omega\tau)$ gives

$$\varepsilon(j\omega)(\sigma + j\omega\tau) = F(j\omega)(\sigma + j\omega\tau) - (\alpha + j\omega\beta) \tag{5.6}$$

Equation (5.6) is treated as a weighted function. The absolute value of equation (5.6) is given by

$$|\varepsilon(\omega)(\sigma + j\omega\tau)|^2 = (R\sigma - I\omega\tau - \alpha)^2 + (R\omega\tau + I\sigma - \omega\beta)^2 \qquad (5.7)$$

E is defined as the function given in (5.7) summed over the sampling frequencies $\omega_k$

thus

$$E = \sum_{k=o}^{m} (R_k\sigma_k - I_k\omega_k\tau_k - \alpha_k)^2 + (R_k\omega_k\tau_k + I_k\sigma_k - \omega_k\beta_k)^2 \qquad (5.8)$$

where

$R_k$ = [magnitude at $\omega_k$] [cos (phase angle at $\omega_k$)]

$I_k$ = [magnitude at $\omega_k$] [sin (phase angle at $\omega_k$)]

The unknown polynomial coefficients $p_i$, $q_i$ are now evaluated on the basis of minimizing the function E. Following the standard mathematical procedures equation (5.8) is differentiated with respect to each of the unknown coefficients $p_i$, $q_i$ and the results set equal to zero.

$$\frac{\partial E}{\partial p_o} = \sum_{k=o}^{m} - 2(\sigma_k R_k - \omega_k\tau_k I_k - \alpha_k) = 0$$

$$\frac{\partial E}{\partial p_1} = \sum_{k=o}^{m} - 2\omega_k(\omega_k\tau_k R_k + \sigma_k - \omega_k\beta_k) = 0 \qquad (5.9)$$

$$\vdots$$

$$\frac{\partial E}{\partial q_1} = \sum_{k=o}^{m} - 2\omega_k I_k(\sigma_k R_k - \omega_k\tau_k I_k - \alpha_k) + 2\omega_k R_k(\omega_k\tau_k R_k + \sigma_k I_k - \omega_k R_k) = 0$$

$$\vdots$$

By the following linear transformations

$$\alpha_k = p_o - \alpha_k'$$

$$\beta_k = p_1 - \beta_k'$$

$$\sigma_k = q_o - \sigma_k' = (1 - \sigma)^*$$

$$\tau_k = q_1 - \tau_k'$$

and substitute the following into the equations,

$$\lambda_i = \sum_{k=o}^{m} \omega_k^i, \qquad S_i = \sum_{k=o}^{m} \omega_k^i R_k$$

$$\tau_i = \sum_{k=o}^{m} \omega_k^i I_k, \qquad U_i = \sum_{k=o}^{m} \omega_k^i (R_k^2 + I_k^2)$$

equation (4.9) becomes

$$p_o\lambda_o - p_2\lambda_2 + p_4\lambda_4 - p_6\lambda_6 + \cdots + q_1\tau_1 + q_2S_2 - q_3\tau_3 - q_4S_4$$
$$+ q_5\tau_5 + q_6S_6 - q_7\tau_7 - \cdots = S_o$$

$$p_1\lambda_2 - p_3\lambda_4 + p_5\lambda_6 - p_7\lambda_8 + \cdots - q_1S_2 + q_2\tau_3 + q_3S_4 - q_4\tau_5$$
$$- q_5S_6 + q_6\tau_7 + q_7S_8 - \cdots = \tau_1$$

$$p_o\lambda_2 - p_2\lambda_4 + p_4\lambda_6 - p_6\lambda_8 + \cdots + q_1\tau_3 + q_2S_4 - q_3\tau_5 - q_4S_6$$
$$\vdots$$
$$+ q_5\tau_7 + q_6S_8 - q_7\tau_9 - \cdots = S_2$$

* For convenience, $q_o$ is set to equal to 1. This does not affect the function in any manner.

$$p_o\tau_1 - p_1S_2 - p_2\tau_3 + p_3S_4 + p_4\tau_5 - \cdots + q_1U_2 - q_3U_4 + q_5U_6 - q_7U_8$$
$$+ q_9U_{10} - q_{11}U_{12} + \cdots = 0$$

$$p_oS_2 + p_1\tau_3 - p_2S_4 - p_3\tau_5 + p_4S_6 + \cdots + q_2U_4 - q_4U_6 + q_6U_8 - q_8U_{10}$$
$$+ q_{10}U_{12} - \cdots = U_2$$

$$p_o\tau_3 - p_1S_4 - p_2\tau_5 + p_3S_6 + p_4\tau_7 - \cdots + q_1U_4 - q_3U_6 + q_5U_8 - q_7U_{10}$$
$$+ q_9U_{12} - \cdots = 0$$
$$\vdots \qquad\qquad \vdots$$

or in matrix notation:

$$(M)(N) = (C) \tag{5.10}$$

i.e

$$\begin{bmatrix} \lambda_o & 0 & -\lambda_2 & 0 & \lambda_4 & \cdots & \tau_1 & S_2 & -\tau_3 & -S_4 & \tau_5 & \cdots \\ 0 & \lambda_2 & 0 & -\lambda_4 & 0 & \cdots & -S_2 & \tau_3 & S_4 & -\tau_5 & -S_6 & \cdots \\ \lambda_2 & 0 & -\lambda_4 & 0 & \lambda_6 & \cdots & \tau_3 & S_4 & -\tau_5 & -S_6 & \tau_7 & \cdots \\ 0 & \lambda_4 & 0 & \tau_6 & 0 & \cdots & -S_4 & \tau_5 & S_6 & -\tau_7 & -S_8 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \tau_1 & -S_2 & -\tau_2 & S_4 & \tau_5 & \cdots & U_2 & 0 & -U_4 & 0 & U_6 & \cdots \\ S_2 & \tau_3 & -S_4 & -\tau_5 & S_6 & \cdots & 0 & U_4 & 0 & -U_6 & 0 & \cdots \\ \tau_3 & -S_4 & -\tau_5 & S_6 & \tau_7 & \cdots & U_4 & 0 & -U_6 & 0 & U_8 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \begin{bmatrix} p_o \\ p_1 \\ p_2 \\ p_3 \\ \vdots \\ q_1 \\ q_2 \\ q_3 \\ \vdots \end{bmatrix} = \begin{bmatrix} S_o \\ \tau_1 \\ S_2 \\ \tau_3 \\ \vdots \\ 0 \\ U_2 \\ 0 \\ \vdots \end{bmatrix} \tag{5.11}$$

The numerical value of the unknown coefficients may thus be obtained from Eq. (5.11).

The method of complex curve-fitting as presented here would correspond to a least-squares fit if $|\sigma + j\omega\tau|$ were a constant. Actually $|\sigma + j\omega\tau|$ is a weighting function; so this method can be described as a "weighted least squares fit". However according to C. K. Sanathanan's opinion (Ref. 10), this "weighted least square fit" has some serious deficiencies. He suggested that the deficiencies can be overcome by an iteration procedure which effectively eliminates the above weighting.

From equation (5.6), let

$$(\sigma + j\omega\tau) = Q(j\omega) \text{ and } (\alpha + j\omega\beta) = P(j\omega)$$

then
$$\varepsilon(\omega)\ Q(j\omega) = F(j\omega)\ Q(j\omega) - p(j\omega)$$

denote

$$\varepsilon(\omega)\ Q(j\omega) \text{ by } \varepsilon_k' \text{ then}$$

$$\varepsilon_k' = F(j\omega_k)\ Q(j\omega_k) - P(j\omega_k) \tag{5.12}$$

equation (5.12) can be modified by writing

$$\varepsilon_k'' = \frac{\varepsilon_k\ Q(j\omega_k)_L}{Q(j\omega_k)_{L-1}} = \frac{F(j\omega_k)\ Q(j\omega_k)_L}{Q(j\omega_k)_{L-1}} - \frac{P(j\omega_k)_L}{Q(j\omega_k)_{L-1}} \tag{5.13}$$

where the subscript L corresponds to the iteration number.

As $Q(j\omega_k)$ is not known initially, it is assumed to be equal to 1. The subsequent iteration tend to converge rapidly and the coefficients evaluated become effectively those obtained by minimizing the sum of $|\varepsilon_k'|^2$ at all the experimental points.

From equation (5.13)

$$|\epsilon_k''|^2 = \frac{|F(j\omega_k)\, Q(j\omega_k)_L - P(j\omega_k)_L|^2}{|Q(j\omega_k)_{L-1}|^2} \tag{5.14}$$

substituting $\omega_{kL} = \dfrac{1}{|Q(j\omega_k)_{L-1}|}$ in equation (5.14) and summing for all k's

$$E = \sum_{k=o}^{M} |\epsilon_k'|^2\, \omega_{kL} \tag{5.15}$$

where $\omega_{kL}$ also equals to

$$\frac{1}{(1-q_{2,L-1}\,\omega^2 + q_{4,L-1}\,\omega^4 - \dots)^2 + \omega^2(q_{1,L-1} - q_{3,L-1}\,\omega^2 + q_{5,L-1}\,\omega^4 - \dots)^2}$$

and $\epsilon_k'$ is a function of $p_o, p_1, p_2 \dots, q_1, q_2, q_3 \dots$ . Now equation (5.15) is partially differentiated with respect to each of the polynomial coefficients, and equal to zero. To evaluate the coefficients, we have the same matrices as in (5.11) except with the following different notations

$$\lambda_i = \sum_{k=1}^{m} \omega_k^i\, \omega_{kL}$$

$$S_i = \sum_{k=1}^{m} \omega_k^i\, R_k\, \omega_{kL}$$

$$T_i = \sum_{k=1}^{m} \omega_k^i\, I_k\, \omega_{kL}$$

$$U_i = \sum_{k=1}^{m} \omega_k\, (R_k^2 + I_k^2)\, \omega_{kL}$$

The coefficients $q_1$, $q_2$, $q_3$, ... evaluated at iteration L-1 are used to evaluate $\omega_L$ for the next iteration.

Due to the elimination of the weighting function $Q(j\omega_k)$ by an iteration procedure, we call this modified method of curve-fitting as "least-square fit".

To define a minimum-phase transfer function within the bounded region of phase and gain specifications, the following computer program has been used. The flow chart is shown in Figure 5-1. It is estimated that the average computer time for all computation in this program is about 6 minutes.

At the present time, the above program is not fully automated because, as it will be seen in section 5.3.1, the desired curve must be drawn before the computer starts the computation. However, with additional study, it is believed that the processes of drawing a desired curve within the bounded region of phase and gain specifications can be programmed into the computer. Thus the whole procedure will be fully automated.

## 5.2 For a Non-Minimum-Phase System

From section 5.1, it has been shown that by using a set of data obtained from experimental results as input to the complex curve fit process, a closely fit transfer function can be obtained through consecutive computer search. Based upon the above principle, a method of defining a transfer function with bounded regions of phase and gain specifications is acquired. In the case of non-minimum-phase system it is so simple that

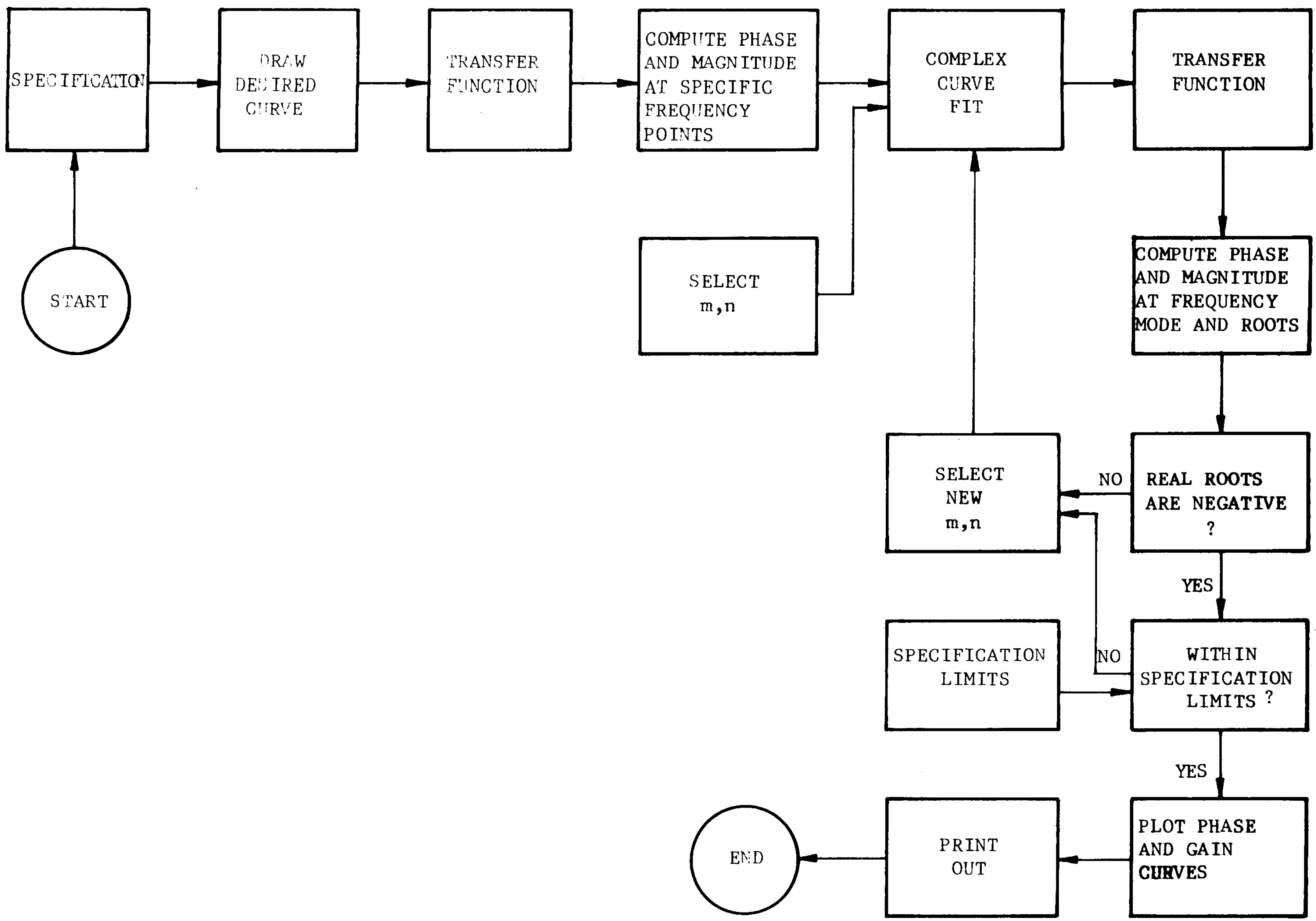



FIGURE 5-1. COMPUTER PROGRAM FLOW CHART

the only thing one has to do is to plot a free hand curve through the bounded regions of phase and gain specifications in log magnitude and phase angle diagram and then pick up, say 30 points, (actually the more the better) from the plotted phase and gain curve within a desired frequency range at equal frequency intervals. After the 30 points have been picked out, one can read off the values of magnitude and phase angle directly from the log magnitude and phase angle diagram at each specific frequency point. Values of frequencies, phase angle and magnitude form a set of data. Using these data as input to the computer program, a closely fit transfer function can be obtained by a successive computer search.

### 5.3 For a Minimum-Phase System

For a minimum-phase system, to obtain a stable transfer function within the bounded regions of phase and gain specification is much more complicated if the phase and gain specifications define only discrete bounded regions. For example, in Figure 5-4, the regions other than those within the rectangles are undefined. If the method in Section 5.2 is applied, more often than not, the transfer functions so obtained will be unstable, although one can always get a good fit.

In order to assure the capability of defining a minimum-phase transfer function within the regions of phase and gain specifications, the minimum-phase requirements and the rules of stability which were discussed in Section 4 must be closely observed when drawing a curve within the bounded regions. This can be accomplished by the application of a set of Bode templates. The techniques will be discussed in the following section.

5.3.1 The Application of Bode Templates

The templates are made in accordance with Figure 4-3 and the second order transfer function

$$F(j\omega) = \frac{1}{(\frac{j\omega}{\omega_n})^2 + 2\zeta \frac{j\omega}{\omega_n} + 1} \tag{5.16a}$$

or

$$F(S) = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2} = \frac{1}{(\frac{S}{\omega_n})^2 + \frac{2\zeta S}{\omega_n} + 1} \tag{5.16b}$$

where

$S = j\omega$ $\qquad$ $\omega$ = frequency (radian per second)

$\omega_n$ = undamped natural (break) frequency

$\zeta$ = damping ratio

$\omega_n$ and $\zeta$ both are constants.

By changing the value of $\zeta$ of (5.16a) a specific gain curve and phase curve can be obtained in the log magnitude and phase diagram with $\frac{\omega}{\omega_n}$ as abscissa. This has been shown in Figure 4-3. To obtain gain curve and phase curve, equation (5.16a) is solved in the following two equations, respectively

$$|F(j\omega)| = \text{magnitude} = \frac{1}{\left[(1 - \frac{\omega^2}{\omega_n^2})^2 + (2\zeta \frac{\omega}{\omega_n})^2\right]^{\frac{1}{2}}}$$

$$\Phi_k = \text{phase angle} = \tan^{-1}\left[\frac{I(\omega_k)}{R(\omega_k)}\right]$$

$$\Phi_k = \tan^{-1}\left[\frac{2\zeta(\frac{\omega}{\omega_n})}{1 - \frac{\omega^2}{\omega_n^2}}\right]$$

A set of 30 templates have been employed for this work in which twenty-nine are derived from eq. (5.16). With damping ratio from .010 through 1.00 and one from equation,

$$\frac{1}{1 + j\frac{\omega}{\omega_n}} = \frac{\omega_n}{\omega_n + S} = \frac{\omega_n}{\omega_n + j\omega}$$

a first order transfer function.

These templates are designed to match 4 cycle semi-log paper; therefore, for convenience it is desirable to use 4 cycle semi-log paper to graph the required transfer function.

The shape of the template with a damping ratio of $\zeta = 0.1$ is shown in Figure 5-2.

To demonstrate the application of these templates, two examples are described as follows:

Example 1, Suppose one wishes to determine a transfer function with damping ratio of 0.1 and break frequency of 6 radian per second and at the same time draw the magnitude and phase diagram. The procedures are very simple. The graph of the curves are shown in Figure 5-3.

Step 1: Mark frequency, magnitude (in db), and phase angle on horizontal and vertical axis. Draw zero db line and zero degree line.

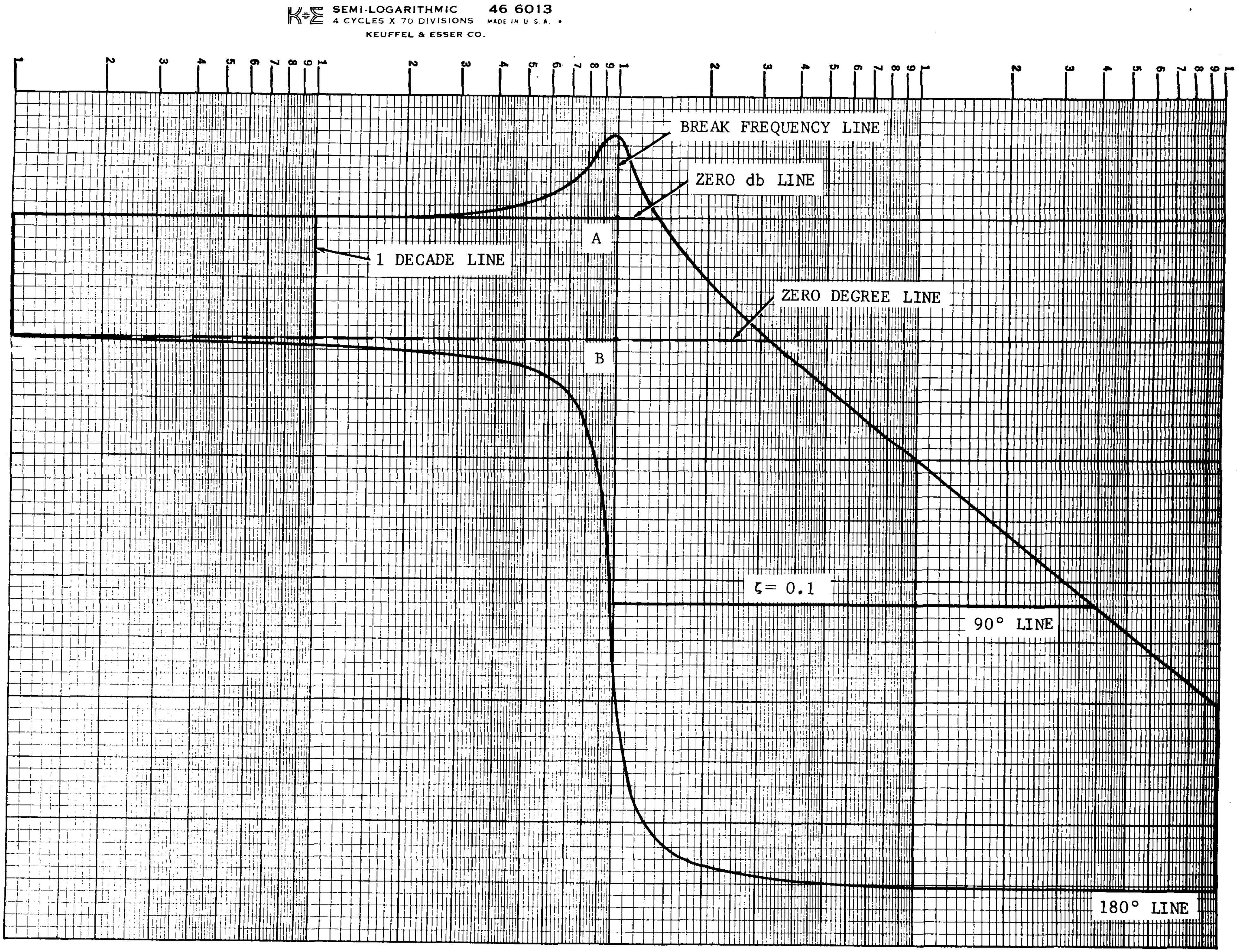



FIGURE 5-2 THE SHAPE OF A TEMPLATE WITH $\zeta = 0.1$

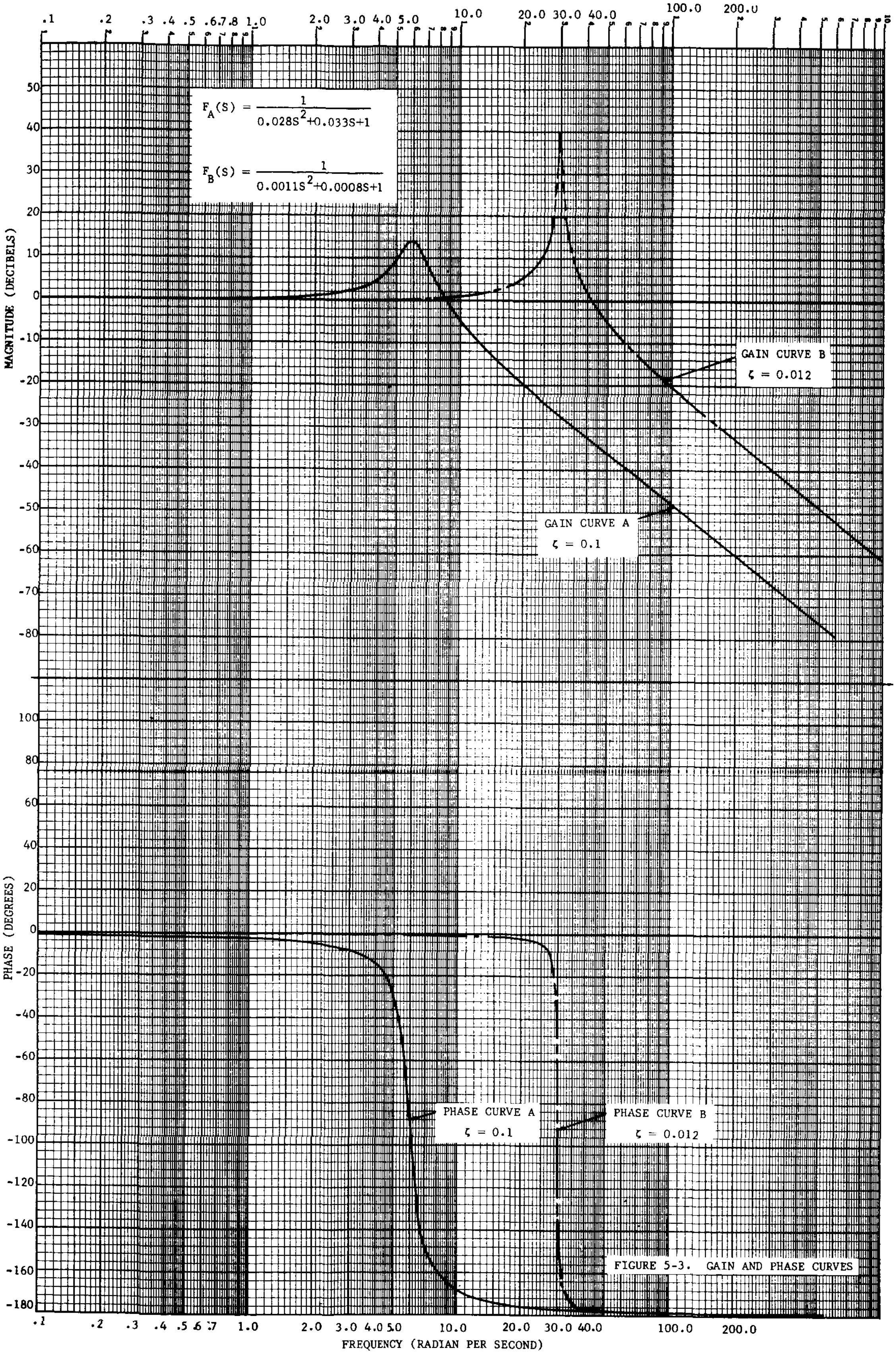


FIGURE 5-3. GAIN AND PHASE CURVES

Step 2: For gain curve lay template of $\zeta = 0.1$ on the semi-log paper make zero db line on template coincident with the zero db line on paper and break frequency line coincident with desired break frequency (in this case $\omega_n = 6$). Then draw gain curve along the upper edge of the template, which is curve A in Fig. 5-3.

Some templates with smaller damping ratio, for example $\zeta = .012$, have a very sharp peak of gain curve. However the sharp peak on template is truncated. In this case, one must complete the peak when drawing the gain curve such as curve B in Figure 5-3.

Step 3: For phase curve, lay the same template ($\zeta = 0.1$) on the lower part of Figure 5-3. Make zero degree line on template coincident with the zero degree line on paper and break frequency line coincident with break frequency ($\omega_n = 6$). Then draw phase curve along the lower edge of the template.

The transfer function for curve A is:

$$F_A(S) = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2}$$

where

$\omega_n = 6$ radian per second, $\zeta = 0.1$

Therefore

$$F_A(S) = \frac{6^2}{S^2 + 2 \times 0.1 \times 6S + 6^2}$$

$$F_A(S) = \frac{1}{0.028S^2 + 0.033S + 1}$$

For curve B:

$$\omega_n = 30 \text{ radian per second}, \qquad \zeta = 0.012$$

Therefore

$$F_B(S) = \frac{1}{0.0011S^2 + 0.0008S + 1}$$

It is clear that one can always obtain different transfer function with the same damping ratio by varying the value of $\omega_n$, the break frequency. One can get a set of curves of the same shape by simply moving the template along the zero db and zero degree lines.

In more complex cases, that is the case in which the polynomials in the transfer function F(S) are of higher order, one can use the combination of more than one template to obtain desired phase and gain curves and transfer functions. To illustrate this point, Example 2 is used.

Example 2: For rate gyro filter specification limits, the total phase and gain curves are defined as shown on Fig. 5-4.

The procedure is as follows:

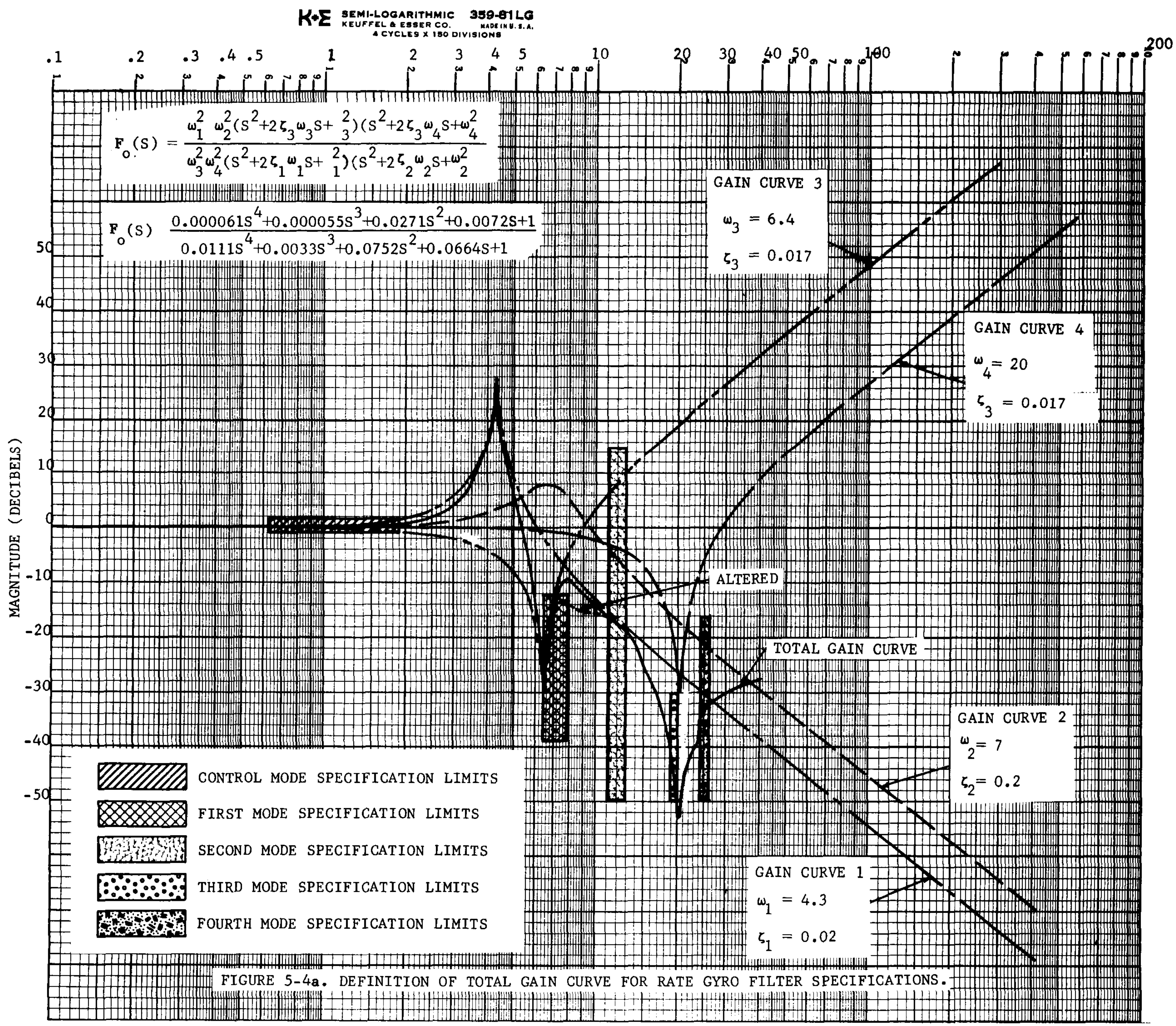



FIGURE 5-4a. DEFINITION OF TOTAL GAIN CURVE FOR RATE GYRO FILTER SPECIFICATIONS.

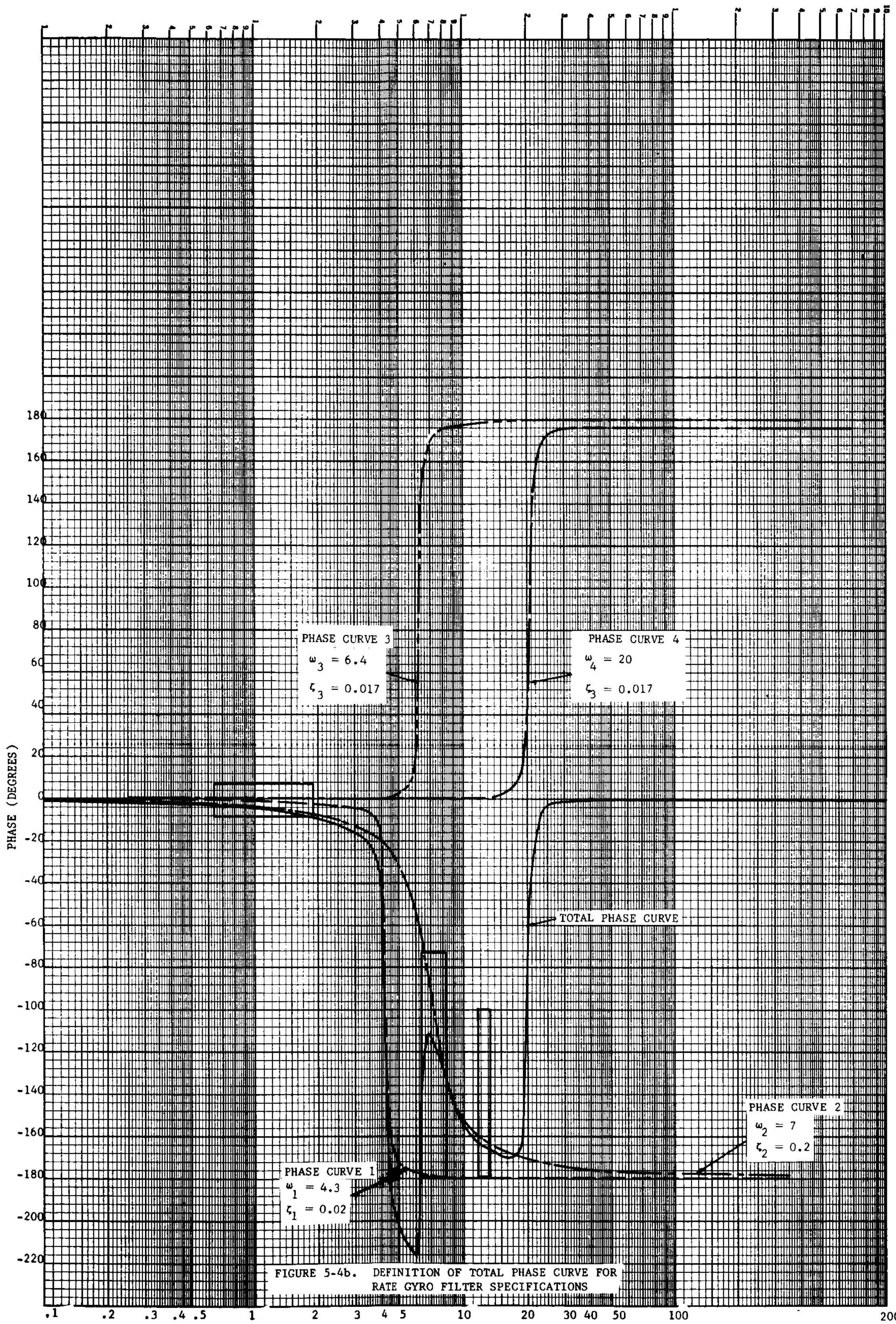



FIGURE 5-4b. DEFINITION OF TOTAL PHASE CURVE FOR RATE GYRO FILTER SPECIFICATIONS

Step 1: Select a set of templates which have slopes that appear consistent with the specification limits and whose summation appears for both total gain and phase curves to be within the specification limits. The above set of templates may be one template or as many as five templates; working with more than five templates becomes unwieldy. In this example, three templates and four different break frequencies have been employed to draw four gain curves and four phase curves. By adding these gain and phase curves at every frequency point, the total gain and phase curves are obtained.

Step 2: Apply method in example 1 to draw curves according to each specific break frequency and template. The templates can be reversed (the peak of gain curve points toward frequency axis). In the former case, the transfer function becomes

$$F(S) = 1 \bigg/ \left(\frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2}\right) = \frac{S^2 + 2\zeta\omega_n S + \omega_n^2}{\omega_n^2}$$

See curve 3 and curves 4 in Figure 5-4 as an example.

Step 3: Obtain total gain and phase curves. Using a pair of dividers this process can be performed easily. In practical application, a computer subroutine is employed to compute and plot the total curves; therefore, it is only necessary to sum up the values of all phase or gain curve at each discrete frequency specification; then check whether the total gain or phase is within the specification limits. If total gain and phase is not within limits, adjust position of curves until summation of phase and gain is within specified limits.

Step 4: Obtain transfer function. For this example Figure 5-4, curve 1 (gain and phase curve) has break frequency $\omega_1 = 4.3$ and damping ratio $\zeta_1 = 0.02$; curve 2 has $\omega_2 = 7$, $\zeta_2 = 0.2$; curve 3, $\omega_3 = 6.4$, $\zeta_3 = 0.017$; curve 4, $\omega_4 = 20$, $\zeta_3 = 0.014$. Since the peak of gain curves 1 and 2 point upward and above zero db line, the transfer function will be

$$F(S) = \frac{\omega_n^2}{S^2 + 2\zeta\omega_n S + \omega_n^2} \tag{5.17}$$

and since the peak of gain curves of 3 and 4 point downward and below the zero db line, the transfer function have the form

$$F(S) = \frac{S^2 + 2\zeta\omega_n S + \omega_n^2}{\omega_n^2} \tag{5.18}$$

Now, we have two transfer functions with the same form as (5.17) and two transfer functions with the same form as (5.18). Multiplying these four transfer functions together a total transfer function is obtained.

$$F_o(S) = \frac{\omega_1^2}{S^2 + 2\zeta_1\omega_1 S + \omega_1^2} \cdot \frac{\omega_2^2}{S^2 + 2\zeta_2\omega_2 S + \omega_2^2} \cdot \frac{S^2 + 2\zeta_3\omega_3 S + \omega_3^2}{\omega_3^2}$$

$$\frac{S^2 + 2\zeta_3\omega_4 + \omega_4^2}{\omega_4^2}$$

$$F_o(S) = \frac{\omega_1^2\omega_2^2\,(S^2 + 2\zeta_3\omega_3 S + \omega_3^2)(S^2 + 2\zeta_3\omega_4 S + \omega_4^2)}{\omega_3^2\omega_4^2\,(\omega_1^2 + 2\zeta_1\omega_1 S + S^2)(\omega_2^2 + 2\zeta_2\omega_2 S + S^2)} \tag{5.19}$$

$$F_o(S) = \frac{\left[(\frac{S}{\omega_3})^2 + \frac{2\zeta_3 S}{\omega_3} + 1\right]\left[(\frac{S}{\omega_4})^2 + \frac{2\zeta_3 S}{\omega_4} + 1\right]}{\left[(\frac{S}{\omega_1})^2 + \frac{2\zeta_1 S}{\omega_1} + 1\right]\left[(\frac{S}{\omega_2})^2 + \frac{2\zeta_2 S}{\omega_2} + 1\right]}$$

Substituting the values of break frequencies and damping ratios into equation (5.19) $F_o(S)$ becomes

$$F_o(S) = \frac{(.02441406S^2 + .00531250S + 1)(.00250000S^2 + .00170000S + 1)}{(.05408328S^2 + .00930233S + 1)(.02040816S^2 + .05714285S + 1)} \tag{5.20a}$$

or

$$F_o(S) = \frac{0.000061S^4 + 0.000055S^3 + 0.0271S^2 + 0.0072S + 1}{0.0111S^4 + 0.0033S^3 + 0.0752S^2 + 0.0664\,S + 1} \tag{5.20b}$$

### 5.3.2 Complex Curve Fit

Equation (5.20) is a minimum-phase transfer function with m = 4, n = 4, (m is the degrees of polynomial in numerator of the transfer functions and n is the degrees of polynomial in the denominator). However, this transfer function is not necessarily the only and simplest transfer function (polynomials with lowest degrees) which meet our requirements. In order to explore every possibility of obtaining a simplest stable transfer function, Eq. (5.20) is programmed into our complex curve fitting process. As can be seen in Figure 5-4, the total gain curve obtained deviates somewhat from the first specification limit. This situation can be corrected by altering the curve as shown in Fig. 5-4 and then programmed the altered results by means of complex curve fitting process. The alteration should be done in such a way that the extent of altering will not cause unstability. This will be discussed further in Section 5.4.2 Case I.

Based on Fig. 5-1, the details of the complex curve fit process is described below:

Take equations (5.20b), the transfer function is fed into the computer, a subroutine will compute data shown in Table 5-1. Input format I is used for the input of transfer function which has the form as equations (5.20b), i.e., $F(S) = \frac{A(S)}{B(S)}$ where A(S) and B(S) are polynomials. Input format II is used for the input of transfer function which has quadratic forms as in equation (5.20a). For our purpose, it is much more convenient to use input format II than format I.

Using transfer function (5.20a) and (5.20b) as example the above Fortran input code sheet are completed. It should be pointed out that, in input format I, A1

FORTRAN INPUT CODE SHEET

```
                      NORTHROP SPACE LABS STEVEN TUNG
111111
0 04 (6E10.0)
6.1       E-55.5    E-52.71   E-27.2    E-31.0    E-0
0 04 (6E10.0)
1.11      E-23.3    E-37.52   E-26.64   E-2.1     E+1

.5        50.       100.
```

MSFC - Form 582 (October 1960)

INPUT FORMAT I

FORTRAN INPUT CODE SHEET

```
             NORTHROP SPACE LABS  STEVEN TUNG
111111
1
.02441406      .00531250      1.00000000
.00250000      .00170000      1.00000000

.05408328      .00930233      1.00000000
.02040816      .05714285      1.00000000

.5        50.       100.
```

MSFC - Form 582 (October 1960)

INPUT FORMAT II

Table 5-1

First Subroutine Computing Results

(also input data for complex curve fit)

| Frequency | | Amplitude (Magnitude) | 20 Log (AMP) (decibel) | Phase -degree |
|---|---|---|---|---|
| ω-rad/sec | f-cycle/sec | | | |
| 0.5 | 0.07961 | 1.00000 | 0.00000 | -2.00000 |
| 1.0 | 0.15920 | 1.00000 | 0.00000 | -4.00000 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 7.0 | 1.11460 | 0.20000(0.158)* | -14.00000(-16.0)* | -122.00000 |
| 7.5 | 1.19420 | 0.31600(0.200 | -10.00000(-14.0) | -120.00000 |
| 8.0 | 1.27380 | 0.34300(0.195) | -9.30000(-14.2) | -126.00000 |
| 8.5 | 1.35350 | 0.28200(0.188) | -11.00000(-14.5) | -134.00000 |
| 9.0 | 1.43312 | 0.23700(0.178) | -12.50000(-15.0) | -141.00000 |
| 9.5 | 1.51273 | 0.21100(0.186) | -13.50000(-15.6) | -144.00000 |
| 10.0 | 1.59235 | 0.20000(0.180) | -14.00000(-15.9) | -152.00000 |
| 10.5 | 1.67197 | 0.16800(0.158) | -15.50000(-16.0) | -157.00000 |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| 40.0 | 6.36943 | 0.04170 | -27.60000 | 0.0000 |

* Figures in the parenthesis are corrections made for bringing the gain curve within limits.

1. The first row of the sheet is for the purpose of case identification. One can write any words on this row concerning case identification.

2. In the second row there are six vertical dashes in the format for curve plotting, the first vertical dash designates the plot of $\omega$ (frequency in radian per second) versus phase angle curve the second dash plots $\omega$ versus db; third, the plot of $\omega$ versus magnitude; fourth, the plot of f (frequency in cycles per second) versus phase angle; fifth, the plot of f versus db; sixth, the plot of f versus magnitude . If one does not need all six curves to be plotted some specific vertical dash or dashes can be eliminated.

3. In row three, "0" in first cell indicates complex polynomial. "04" in 3rd and 4th cells means the degree of the complex polynomial and (6E10.0) in the cells from 6 to 13 is the format statement for the coefficients of polynomial in which 10.0 indicates that 10 cells will be used in the following row for each polynomial coefficient.

4. In the fourth row, the value of coefficients of polynomial are put in exponential form. For example $0.000061 = 6.1 \times 10^{-5}$. This value will be shown in the fourth row as 6.1 E-5. The coefficient of highest degree should be put in first 10 cells i.e., those cells from 1 to 10, then the coefficients of lower degree should be put in next 10 cells i.e., from 11th to 20th cell, etc.

5. The usage of rows three and four is the same as row five and six. Third and fourth rows are used for the numerator of the transfer functions and the fifth and sixth rows for the denominator of the transfer function.

6. Row 8 is used for frequency range and number of points desired. .5 and 50. in cells 1 - 2 and 11 - 13 indicate the frequency range from .5 to 50 rad./sec. 100. in cells 21 - 24 means that 100 points within the above frequency range must be computed.

The input format II, rows 1, 2 and 10 have the same usage as rows 1, 2, and 8 in input format I. Other rows are used to designate the following:

1. In row 3 "1" in the first cell indicates complex quadratic forms.

2. Row 4 and 5 are used for the coefficients of quadratic forms in the numerator of the transfer function. In the space from cell 1 to 15, the coefficient of $S^2$ term of the first quadratic will be utilized. In cells from 16 to 30 will be the coefficient of S and from 31 to 45 the constant term 1. The coefficients of the next quadratic will be in row 5. Row 7 and row 8 are for the coefficients of quadratic forms in the denominator of the transfer functions.

Next, another subroutine will plot phases and gain curves in the log magnitude and phase diagram for the transfer function which is shown in Figure 5-5 (Solid Lines). Using the above data as input, computer continues complex curve fit process, searching for a simplest and stable transfer function within the specification limits through a computational loop shown in Fig. 5-1, by changing the degree of polynomials m and n. A sequency of $\frac{m}{n}$ has been programmed into the computer, i.e., $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{3}$, $\frac{3}{3}$, $\frac{1}{4}$, $\frac{2}{4}$, $\frac{3}{4}$, $\frac{4}{4}$, ... $\frac{m}{n}$. The computer can either compute starting from the simplest $\frac{m}{n}$ (i.e., $\frac{1}{2}$) and following the sequence one by one up to higher degree or compute only particular ones ordered by the user. The computation process will be stopped automatically as soon as the computer finds a simplest desirable transfer function. In this example, seven computations ($\frac{m=2}{n=2}$, $\frac{2}{3}$, $\frac{2}{4}$, $\frac{3}{4}$, $\frac{4}{4}$, $\frac{3}{5}$, $\frac{4}{5}$,) have been made in which four of the seven transfer functions are stable and only two of them are within the specification limits. The results of each computation will include a set of data the same as those in Table 5-1, a transfer function and the zeros and poles (all are the roots of the complex polynomials) of the transfer function. Finally the subroutine will plot the phase and gain curves for those desirable transfer functions.

The two desirable transfer functions, their zeros and poles and the phase and gain curves are shown in Figure 5-5 and 5-6.

It has been pointed out that the gain curve in Figure 5-4 is somewhat outside the upper specification limit at the first frequency mode. Figure 5-5a and 5-6a indicates that the fitted gain curves (dotted line) is approximately the same as the original one (solid line). These results are not very satisfactory. Thus the gain curve in Figure 5-4 i.e., the original

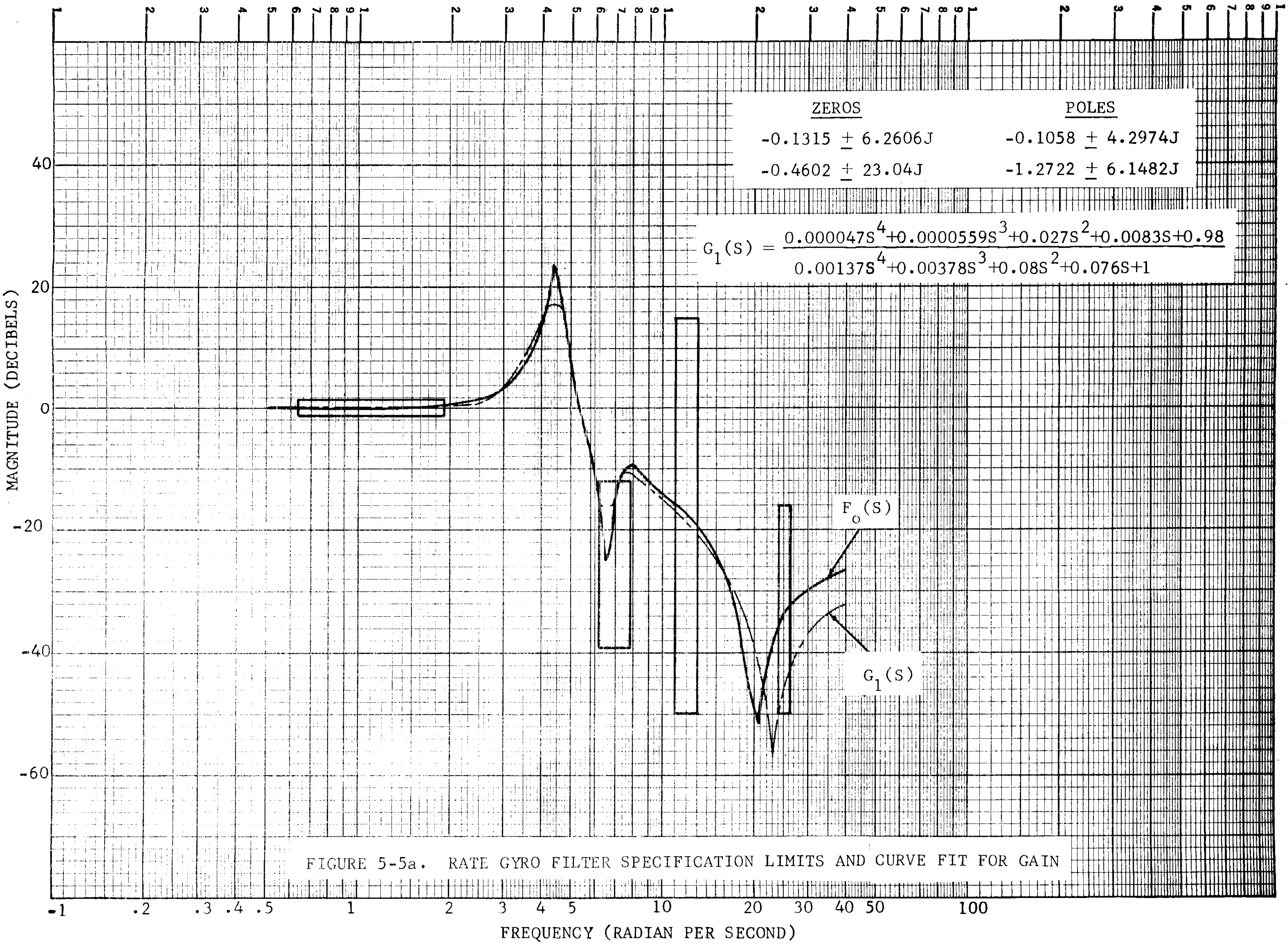



FIGURE 5-5a. RATE GYRO FILTER SPECIFICATION LIMITS AND CURVE FIT FOR GAIN


FIGURE 5-5b. RATE GYRO FILTER SPECIFICATION LIMITS AND CURVE FIT FOR PHASE

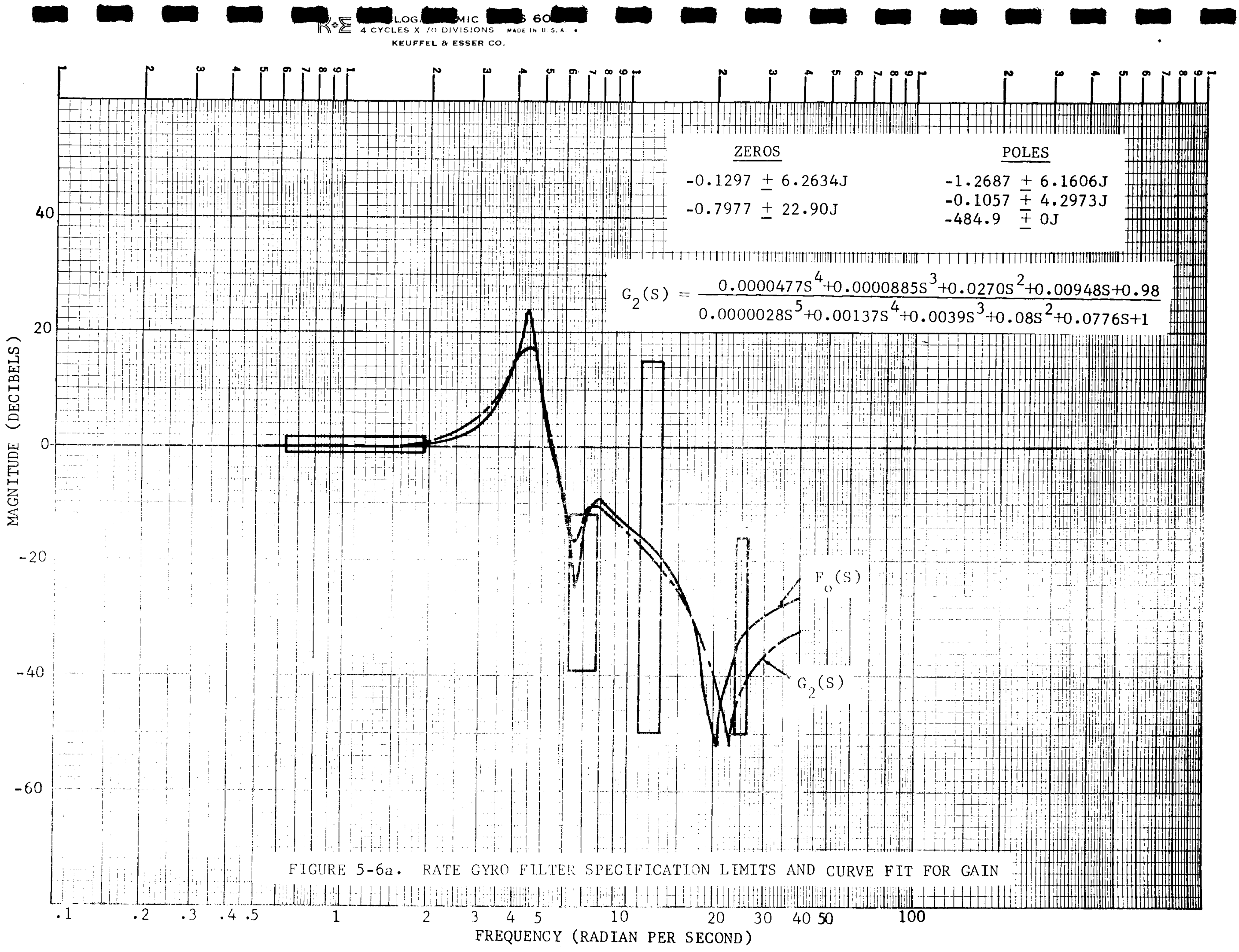



FIGURE 5-6a. RATE GYRO FILTER SPECIFICATION LIMITS AND CURVE FIT FOR GAIN

K+E SEMI-LOGARITHMIC 46 6013
4 CYCLES X 70 DIVISIONS MADE IN U.S.A.
KEUFFEL & ESSER CO.

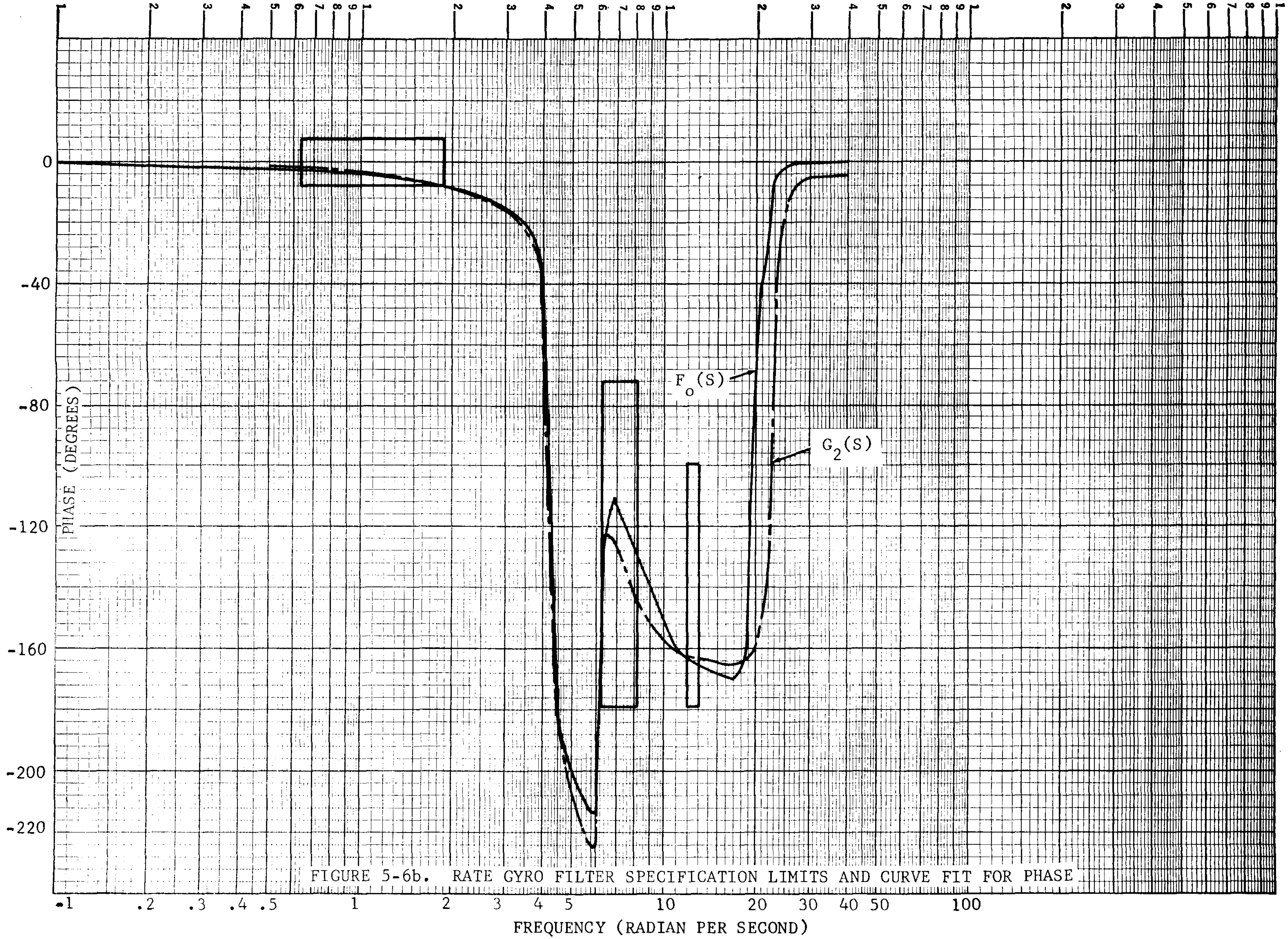



FIGURE 5-6b. RATE GYRO FILTER SPECIFICATION LIMITS AND CURVE FIT FOR PHASE

gain curve (solid line) in Figure 5-7a is modified in order to bring the section of the curve within the upper limit. The value of magnitude at each frequency point is read out as shown in the parenthesis of Table 5-1. Using these modified data as input, through curve fit process, a new set of computations results are shown in Figure 5-7 and 5-8. Now the fitted gain curves in Figure 5-7a and 5-7b are well within the specification limits. The simplest transfer function obtained is:

$$G_3(S) = \frac{0.000054S^4 + 0.000201S^3 + 0.025944S^2 + 0.018766S + 0.957323}{0.001508S^4 + 0.004120S^3 + 0.082812S^2 + 0.0817895S + 1} \qquad (5.21)$$

## 5.4 Study Results

Two conditions have been studied. The results of the study are considered very satisfactory.

### 5.4.1 Condition I, The Bounded Regions of Phase and Gain Specifications are Well Defined:

In this case, the bounded regions of phase or gain specifications is well defined, i.e., the bounded region is analytic and stable. For this situation the definition of a stable transfer function within the bounded regions is very simple and the free-hand curve plot method discussed in Section 5.2 can be utilized.

5.4.1.1 Case 1: Saturn SA-9 Roll Attitude Rate Shaping Network Frequency Response S-I Stage Burn. Given two well defined bounded regions as shown in Figure 5-9, one can draw phase and gain curves by free hand, well within the

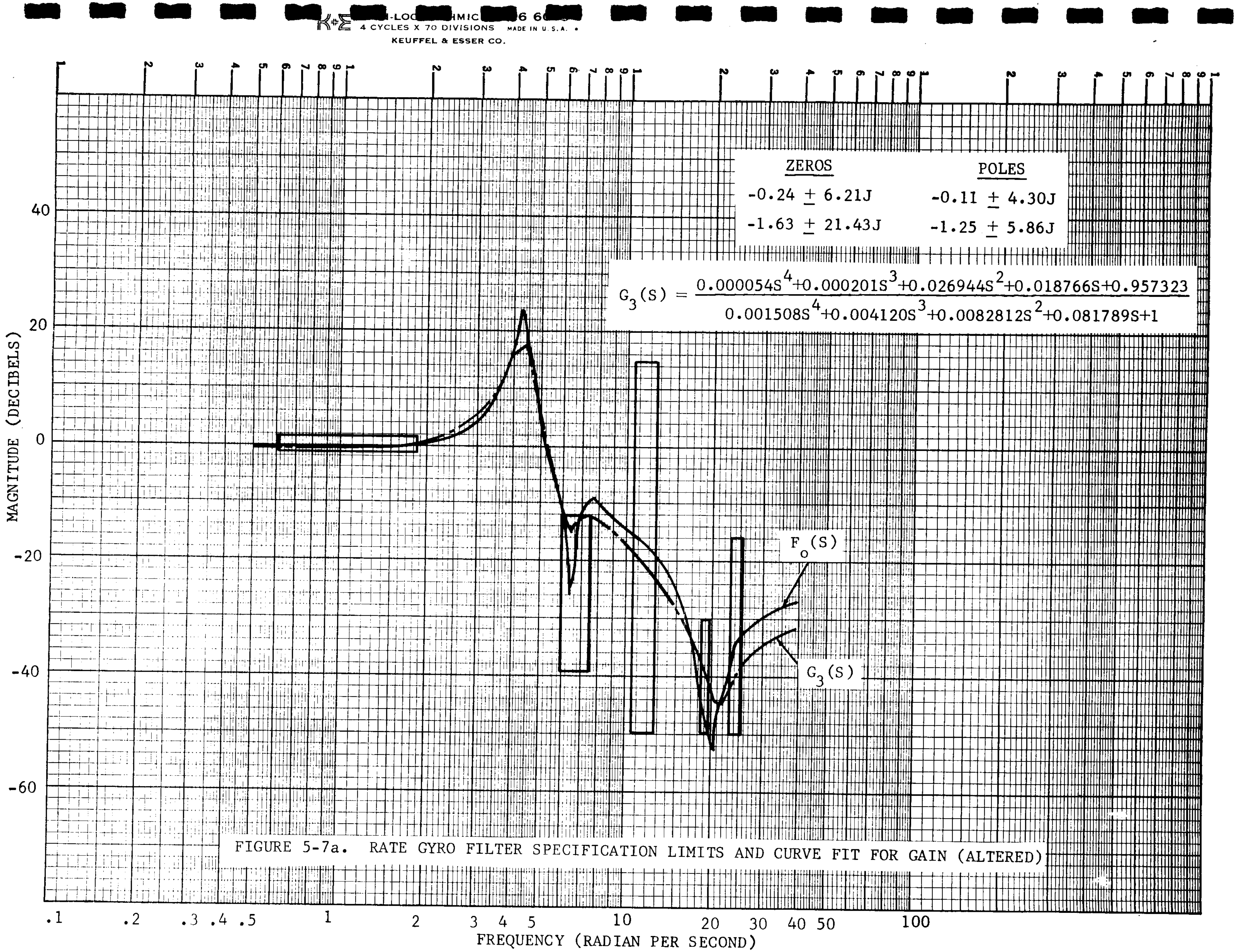



FIGURE 5-7a. RATE GYRO FILTER SPECIFICATION LIMITS AND CURVE FIT FOR GAIN (ALTERED)

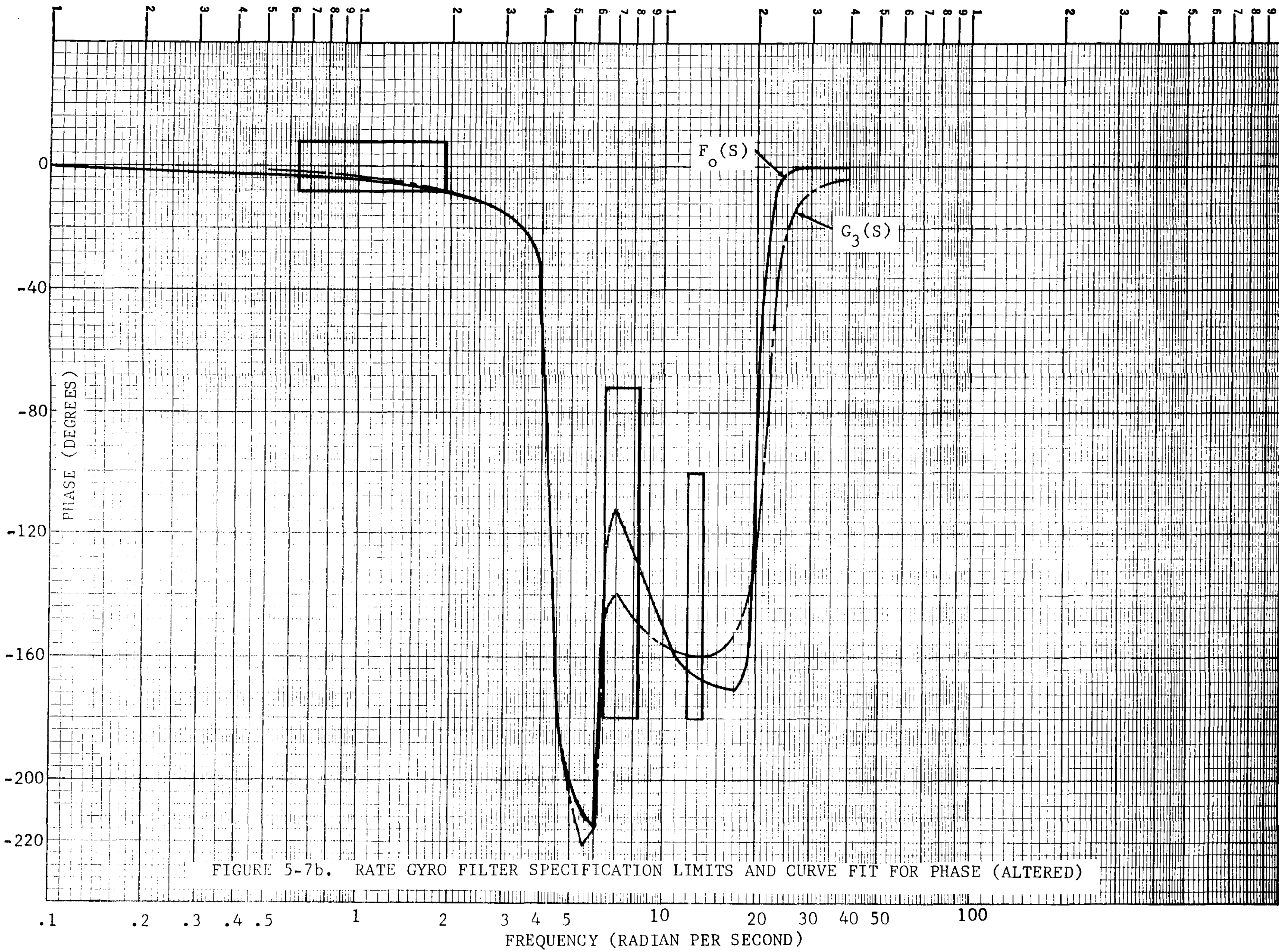



FIGURE 5-7b. RATE GYRO FILTER SPECIFICATION LIMITS AND CURVE FIT FOR PHASE (ALTERED)

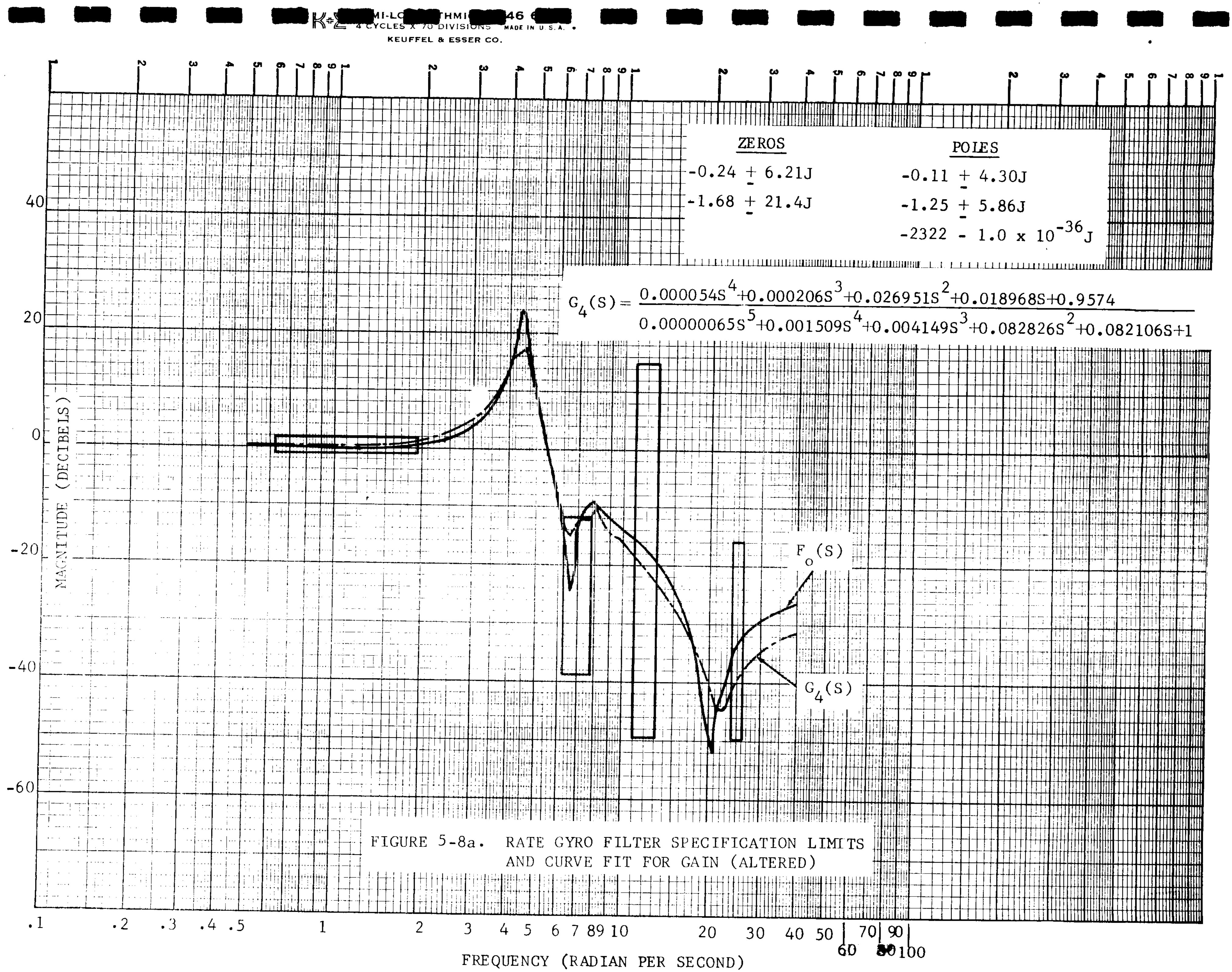



FIGURE 5-8a. RATE GYRO FILTER SPECIFICATION LIMITS AND CURVE FIT FOR GAIN (ALTERED)

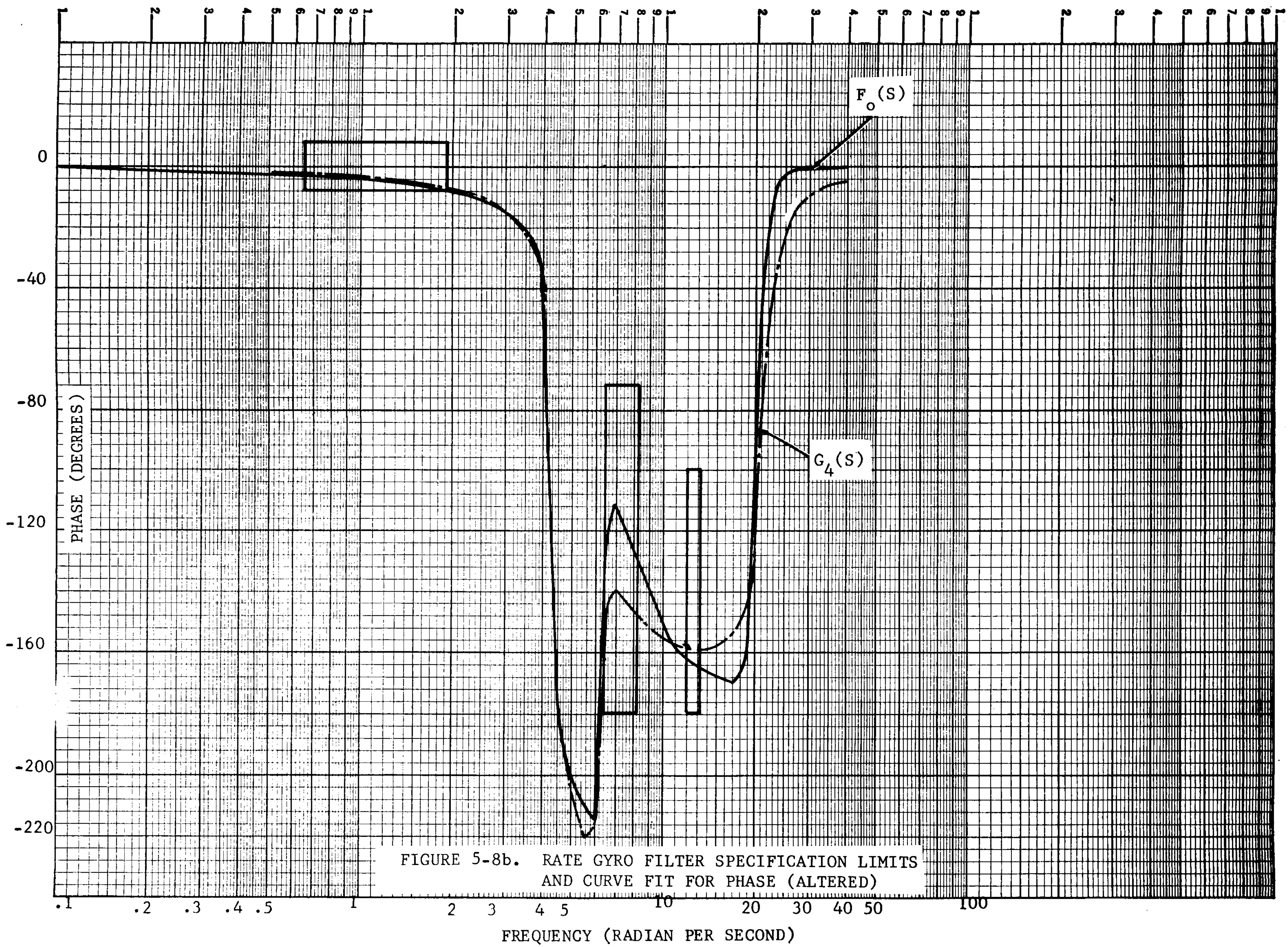



FIGURE 5-8b. RATE GYRO FILTER SPECIFICATION LIMITS AND CURVE FIT FOR PHASE (ALTERED)

$$G_1(S) = \frac{0.000168S^3+0.00160S^2+0.0141S+0.988}{0.0000000159S^5+0.0000088S^4+0.000522S^3+0.0116S^2+0.13S+1}$$

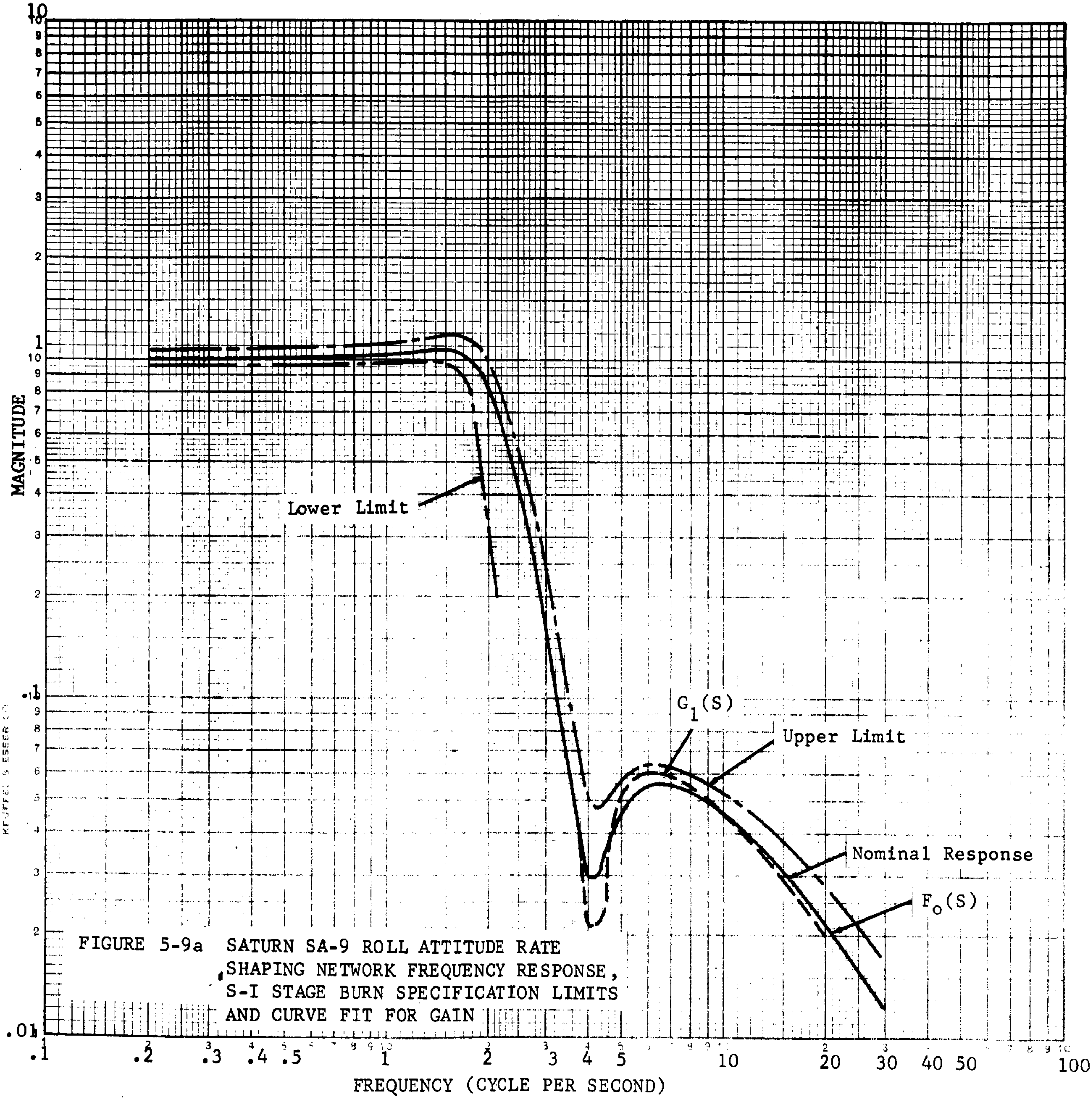



FIGURE 5-9a SATURN SA-9 ROLL ATTITUDE RATE SHAPING NETWORK FREQUENCY RESPONSE, S-I STAGE BURN SPECIFICATION LIMITS AND CURVE FIT FOR GAIN

bounded regions as nominal response. Then pick up some 80 points from each phase and gain nominal response curves within a desired frequency range (in this case, the frequency range is 0.2 - 29.8 rad/sec) at every equal frequency interval. Using the values of frequency, phase angle, and magnitude at each point as input of complex curve fitting process four minimum-phase transfer functions were obtained. The transfer function and their roots are shown in equations (5.22), (5.23), (5.24) and (5.25). The fitted phase and gain curves shown in Figure (5-9), (5-10), (5-11) and (5-12) are well within the specification limits.

$$G_1(S) = \frac{0.000017S^3 + 0.001600S^2 + 0.014100S + 0.988000}{0.0000000159S^5 + 0.0000088S^4 + 0.000522S^3 + 0.0116S^2 + 0.130S + 1} \tag{5.22}$$

| Zeros | Poles |
|---|---|
| $-93.04 + 0j$ | $-27.24 \pm 5.28j$ |
| $-1.12 \pm 25.14j$ | $-4.79 \pm 11.99j$ |
| | $-490.35 - 1.27 \times 10^{-21}j$ |

$$G_2(S) = \frac{0.00149S^2 + 0.00467S + 0.99}{0.00000279S^4 + 0.000439S^3 + 0.01S^2 + 0.123S + 1} \tag{5.23}$$

| Zeros | Poles |
|---|---|
| $-1.57 \pm 25.76j$ | $-16.46 - 1.93 \times 10^{-33}j$ |
| | $-4.51 \pm 12.04j$ |
| | $-131.91 + 0j$ |

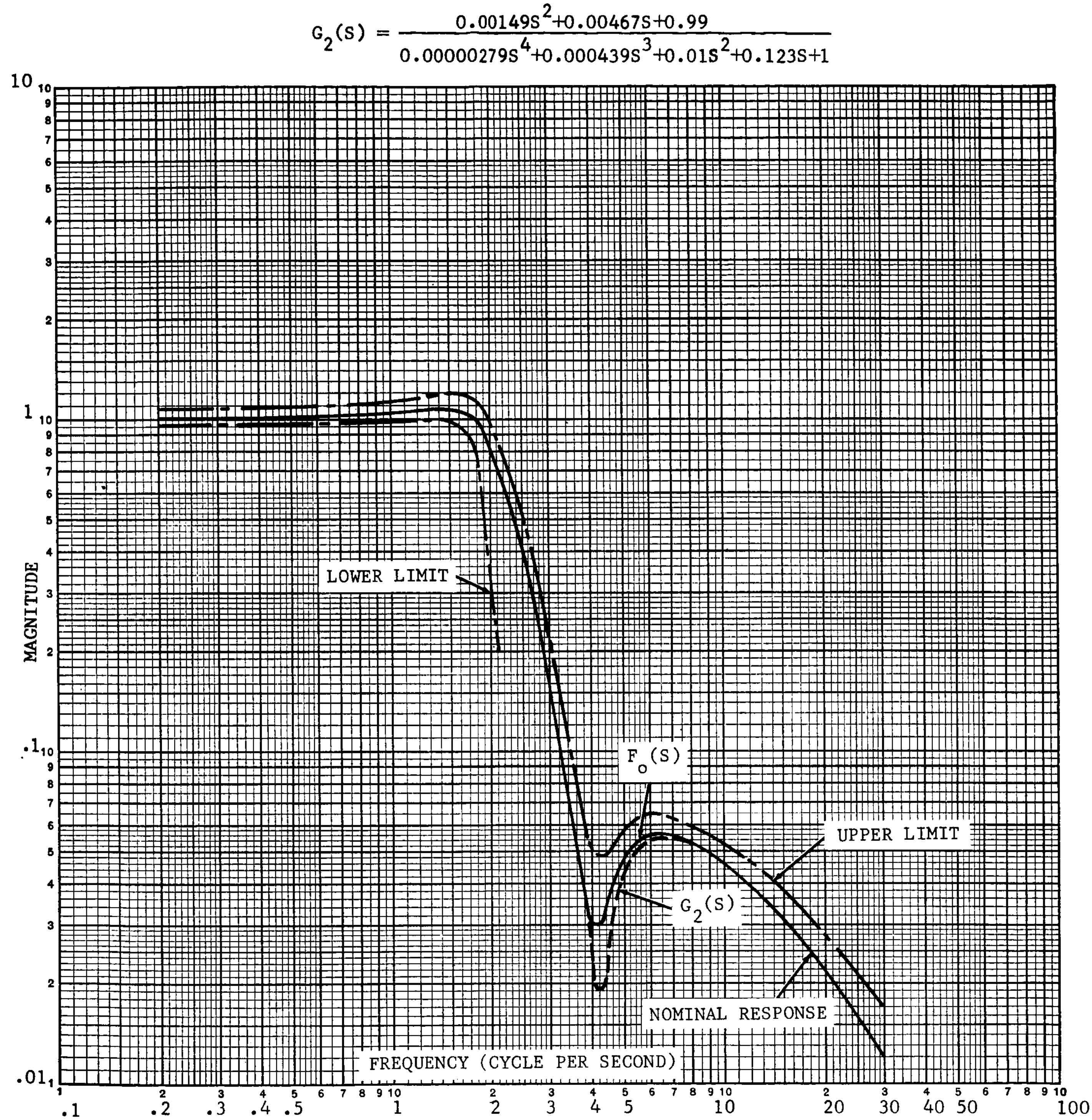



FIGURE 5-10a. SATURN SA-9 ROLL ATTITUDE RATE SHAPING NETWORK FREQUENCY RESPONSE, S-I STAGE BURN SPECIFICATION LIMITS AND CURVE FIT FOR GAIN

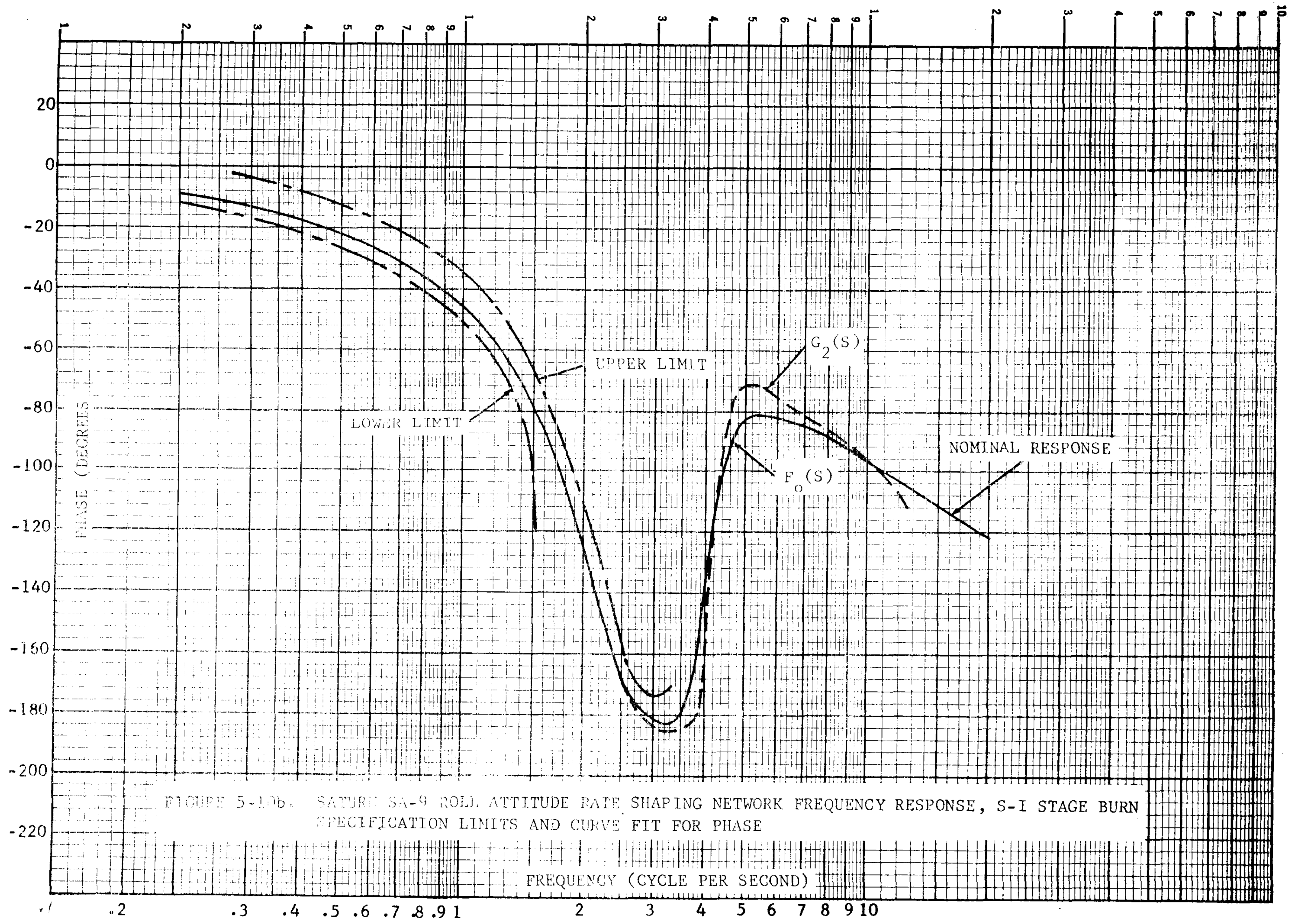



FIGURE 5-10b. SATURN SA-9 ROLL ATTITUDE RATE SHAPING NETWORK FREQUENCY RESPONSE, S-I STAGE BURN SPECIFICATION LIMITS AND CURVE FIT FOR PHASE

$$G_3(S) = \frac{0.00000608S^3+0.00155S^2+0.0076S+0.99}{0.00000509S^4+0.000460S^3+0.0107S^2+0.124S+1}$$

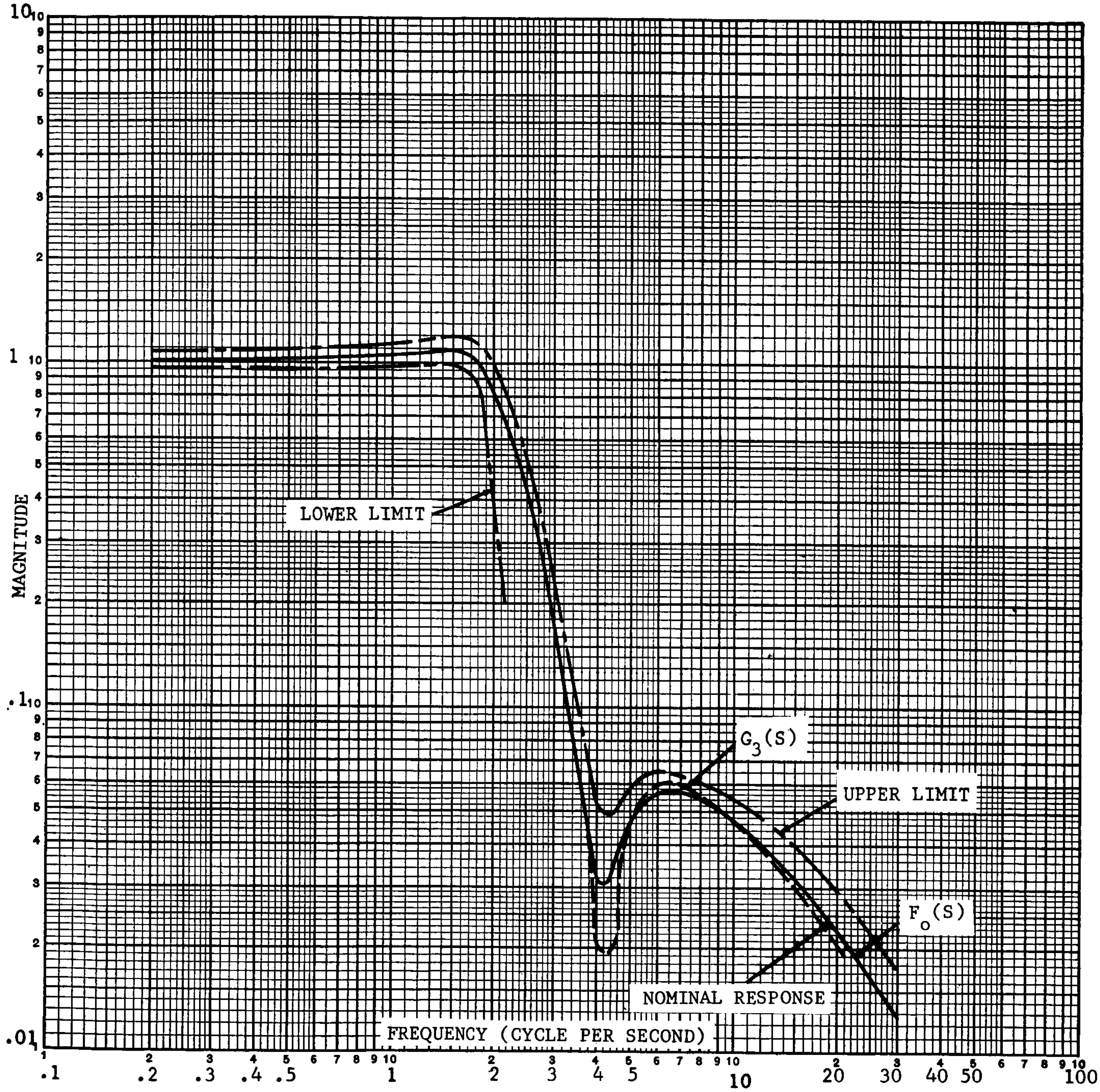



**FIGURE 5-11a. SATURN SA-9 ROLL ATTITUDE RATE SHAPING NETWORK FREQUENCY RESPONSE,** S-I STAGE BURN SPECIFICATION LIMITS AND CURVE FIT FOR GAIN

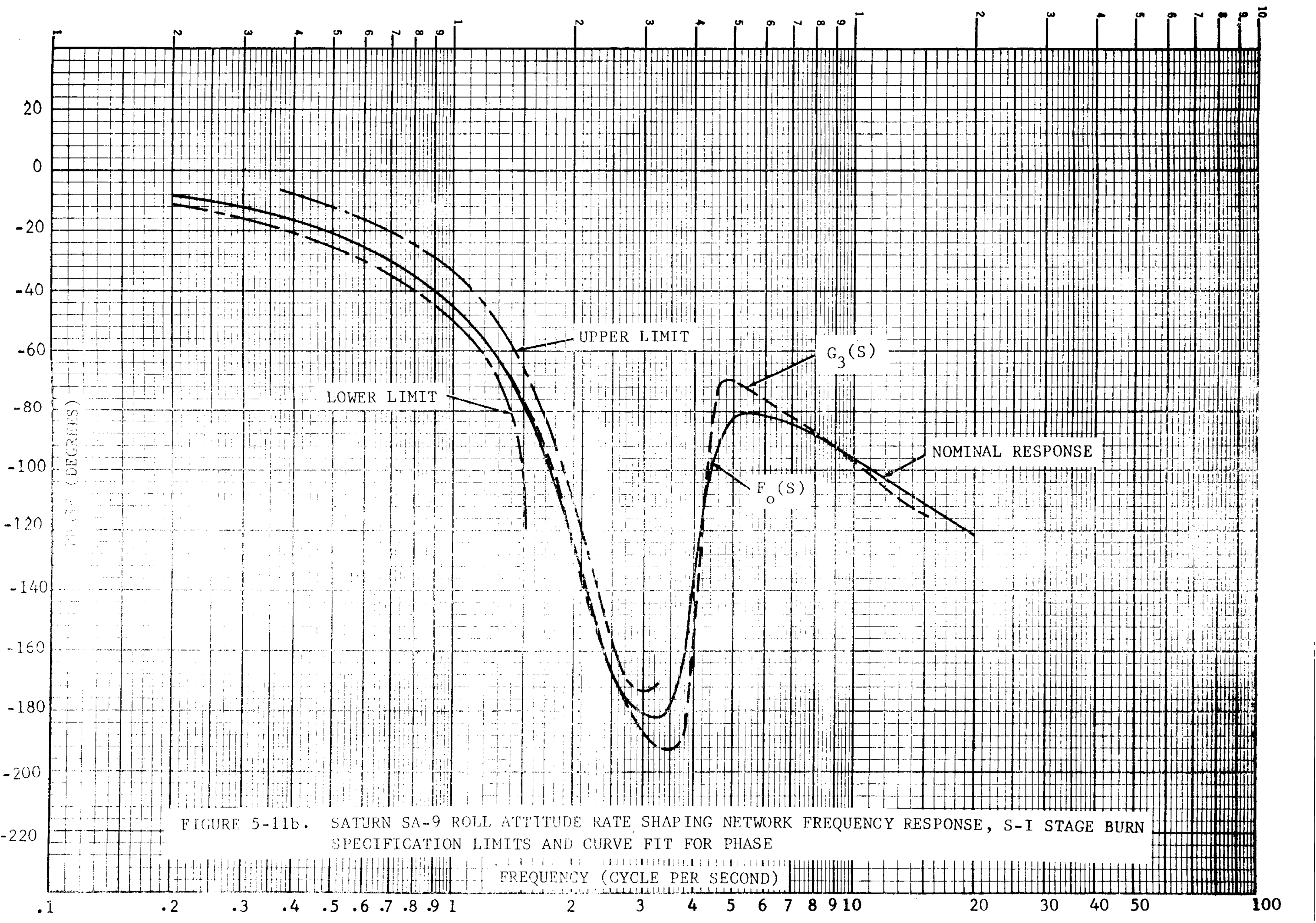



FIGURE 5-11b. SATURN SA-9 ROLL ATTITUDE RATE SHAPING NETWORK FREQUENCY RESPONSE, S-I STAGE BURN SPECIFICATION LIMITS AND CURVE FIT FOR PHASE

$$G_4(s) = \frac{0.00139s^2+0.0032s+0.997}{0.000437s^3+0.0099s^2+0.124s+1}$$

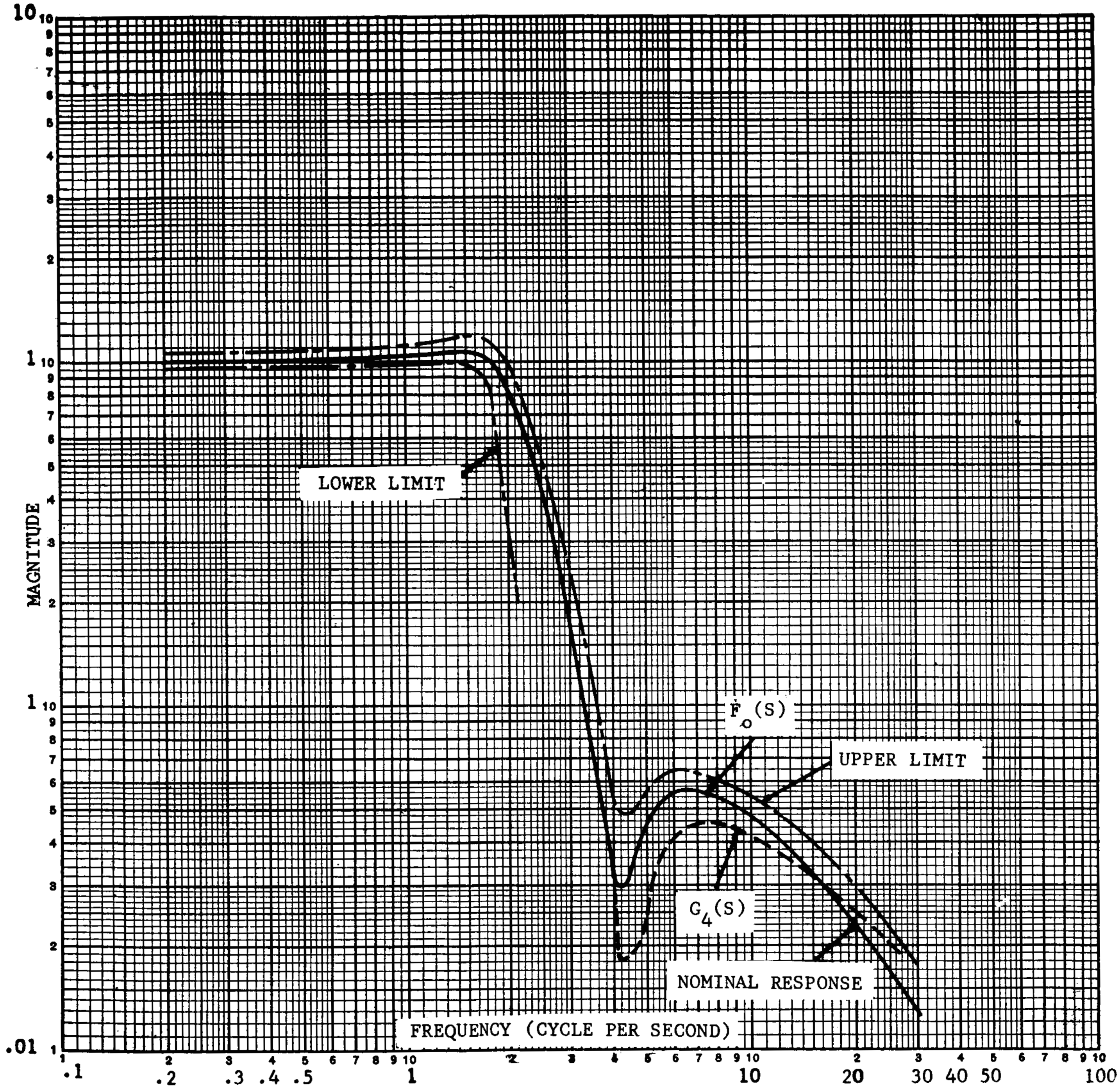
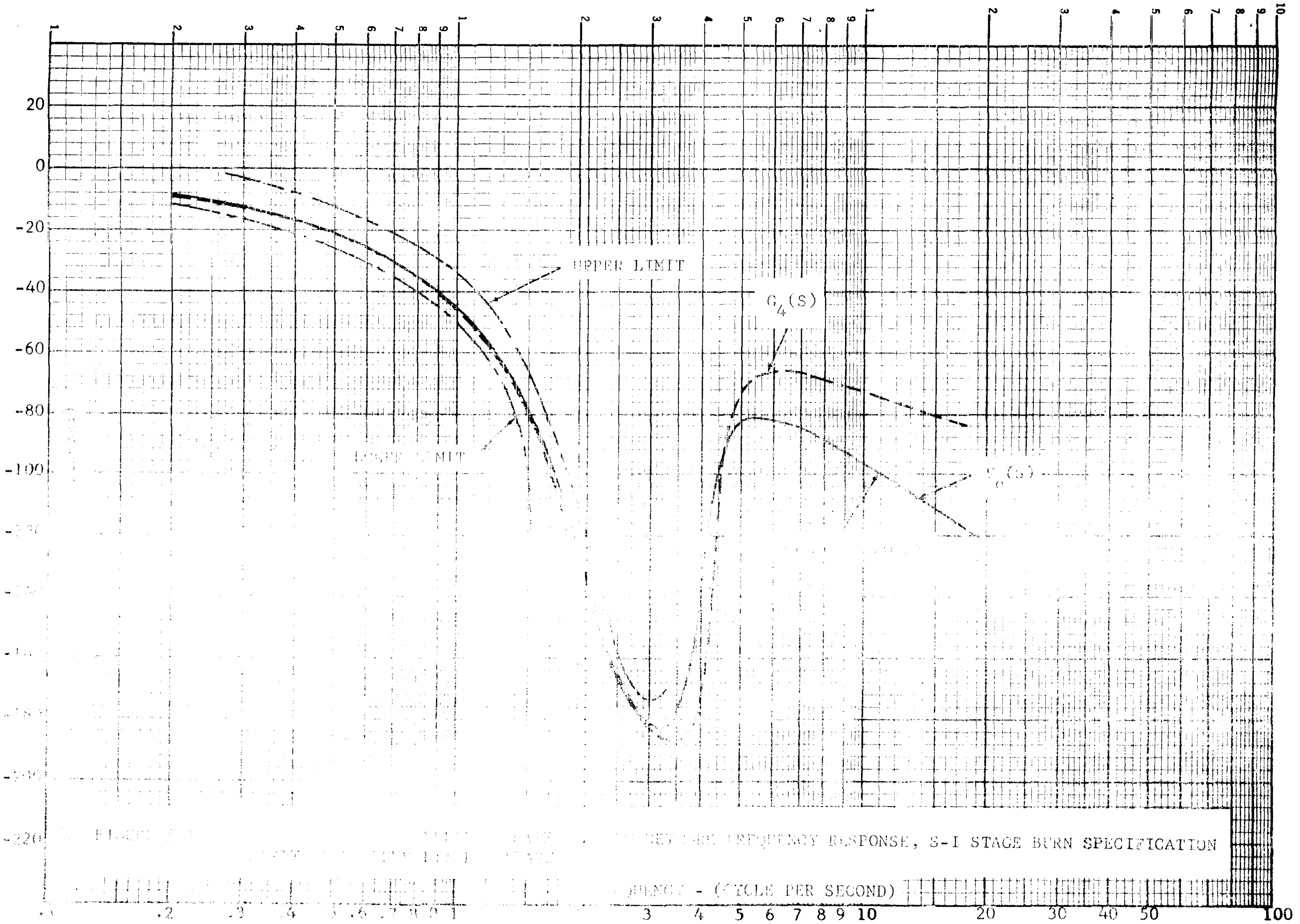



**FIGURE 5-12a. SATURN SA-9 ROLL ATTITUDE RATE SHAPING NETWORK FREQUENCY RESPONSE,**
S-I STAGE BURN SPECIFICATIONS LIMITS AND CURVE FIT FOR GAIN


UPPER LIMIT
LOWER LIMIT
$G_4(S)$
$G_o(S)$
RESPONSE, S-I STAGE BURN SPECIFICATION
(CYCLE PER SECOND)
20
0
-20
-40
-60
-80
-100
-220
.2
.3
.4
1
3
4
5
6
7
8
9
10
20
30
40
50
100

$$G_3(S) = \frac{0.00000608S^3 + 0.00155S^2 + 0.0076S + 0.99}{0.00000509S^4 + 0.00046S^3 + 0.0107S^2 + 0.124S + 1} \tag{5.24}$$

| Zeros | Poles |
|---|---|
| $-1.20 \pm 25.39j$ | $-19.14 + 0j$ |
| $-252.11 - 2.82 \times 10^{-37}j$ | $-4.65 \pm 12.01j$ |
| | $-61.89 + 0j$ |

$$G_4(S) = \frac{0.00139S^2 + 0.0052S + 0.997}{0.000437S^3 + 0.0099S^2 + 0.124S + 1} \tag{5.25}$$

| Zeros | Poles |
|---|---|
| $-1.87 \pm 26.72j$ | $-14.12 + 0j$ |
| | $-4.30 \pm 11.98j$ |

It can be seen that transfer function (5.25) is the most simple one as far as the degrees of polynomial are concerned.

5.4.1.2 Case 2: Saturn SA-9 Pitch-Yaw Attitude Rate Shaping Network Frequency Response S-I Stage Burn. The method used to define a transfer function is exactly the same as Case 1. Two stable transfer functions (see eq.'s 5.26 and 5.27) were obtained. The fitted phase and gain curves shown in Figures (5.13) and (5.14) are well within the specification limits.

$$G_1(S) = \frac{0.0000064S^4 + 0.000064S^3 + 0.00576S^2 + 0.028S + 1}{0.00000109S^6 + 0.00000318S^5 + 0.0000669S^4 + 0.0019S^3 + 0.0239S^2 + 0.182S + 1} \tag{5.26}$$

$$G_1(S) = \frac{0.0000064S^4+0.000064S^3+0.00576S^2+0.028S+1}{0.00000109S^6+0.00000318S^5+0.0000669S^4+0.0019S^3+0.0239S^2+0.182S+1}$$

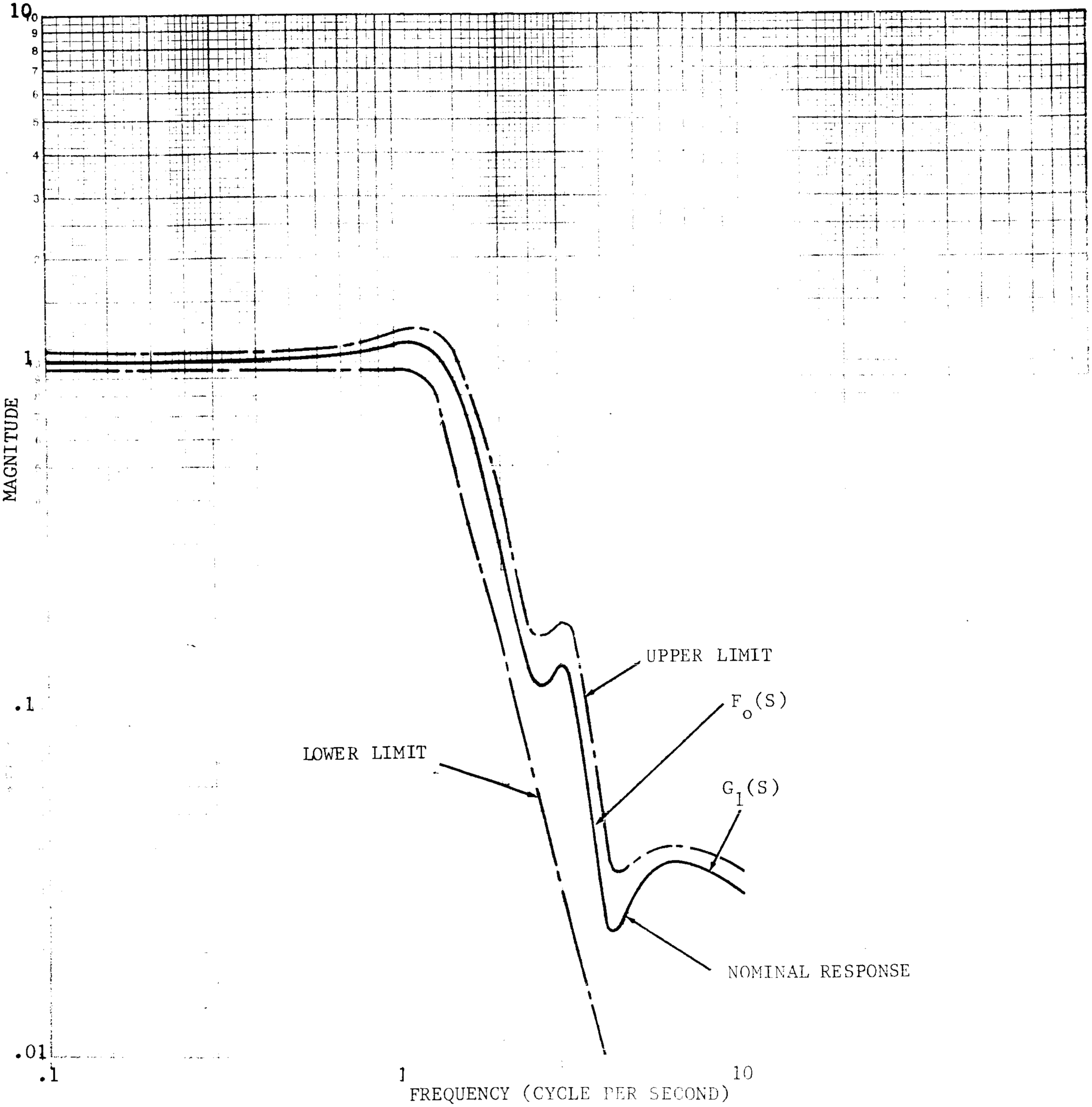



FIGURE 5-13a SATURN SA-9 PITCH-YAW ATTITUDE RATE SHAPING NETWORK FREQUENCY RESPONSE, S-1 STAGE BURN SPECIFICATION LIMITS AND CURVE FIT FOR GAIN

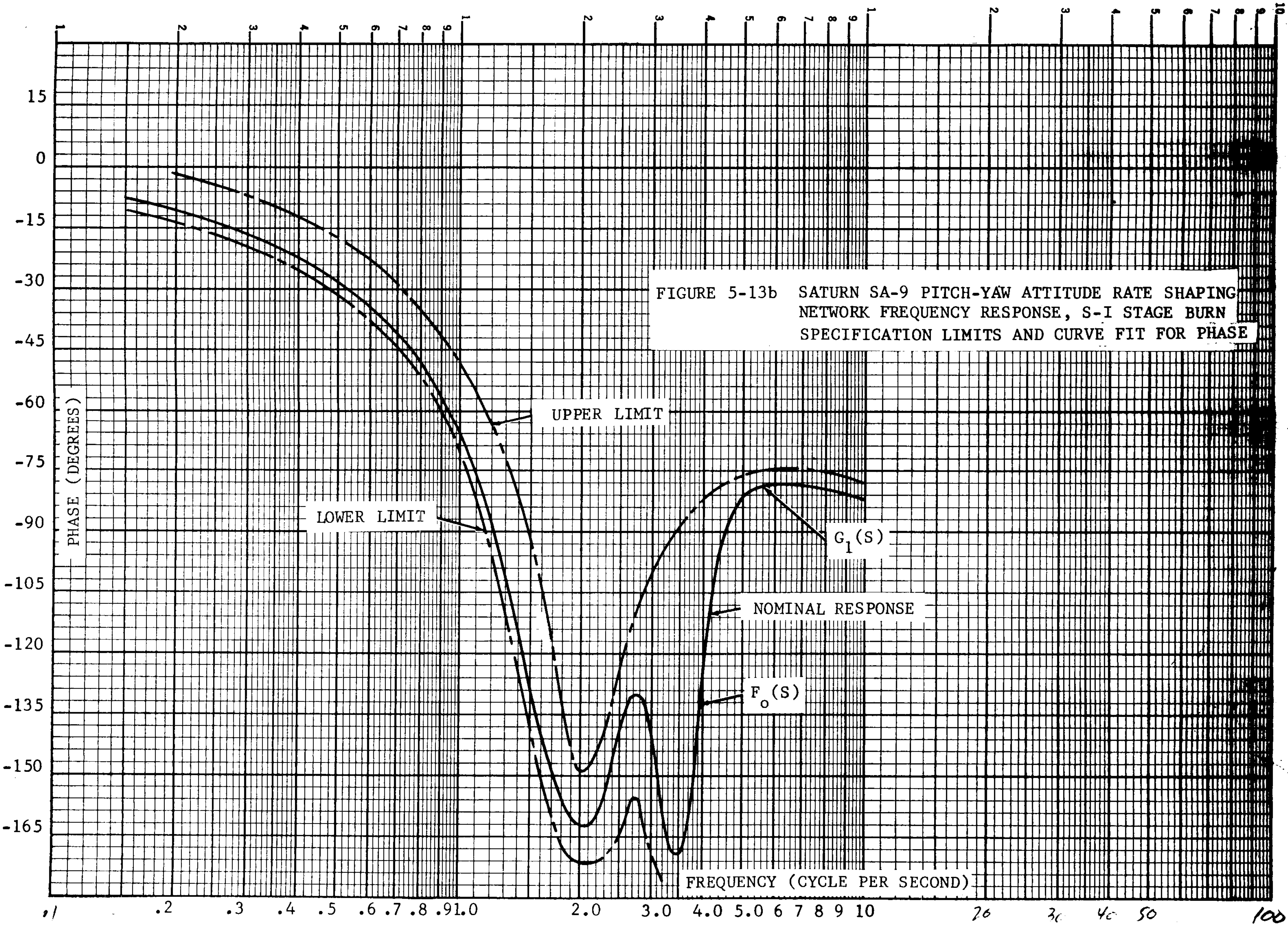



FIGURE 5-13b SATURN SA-9 PITCH-YAW ATTITUDE RATE SHAPING NETWORK FREQUENCY RESPONSE, S-I STAGE BURN SPECIFICATION LIMITS AND CURVE FIT FOR PHASE

$$G_2(S) = \frac{0.0000066S^4+0.0000725S^3+0.00605S^2+0.00298S+1.03}{0.00000298S^5+0.0000707S^4+0.00182S^3+0.00245S^2+0.175S+1}$$

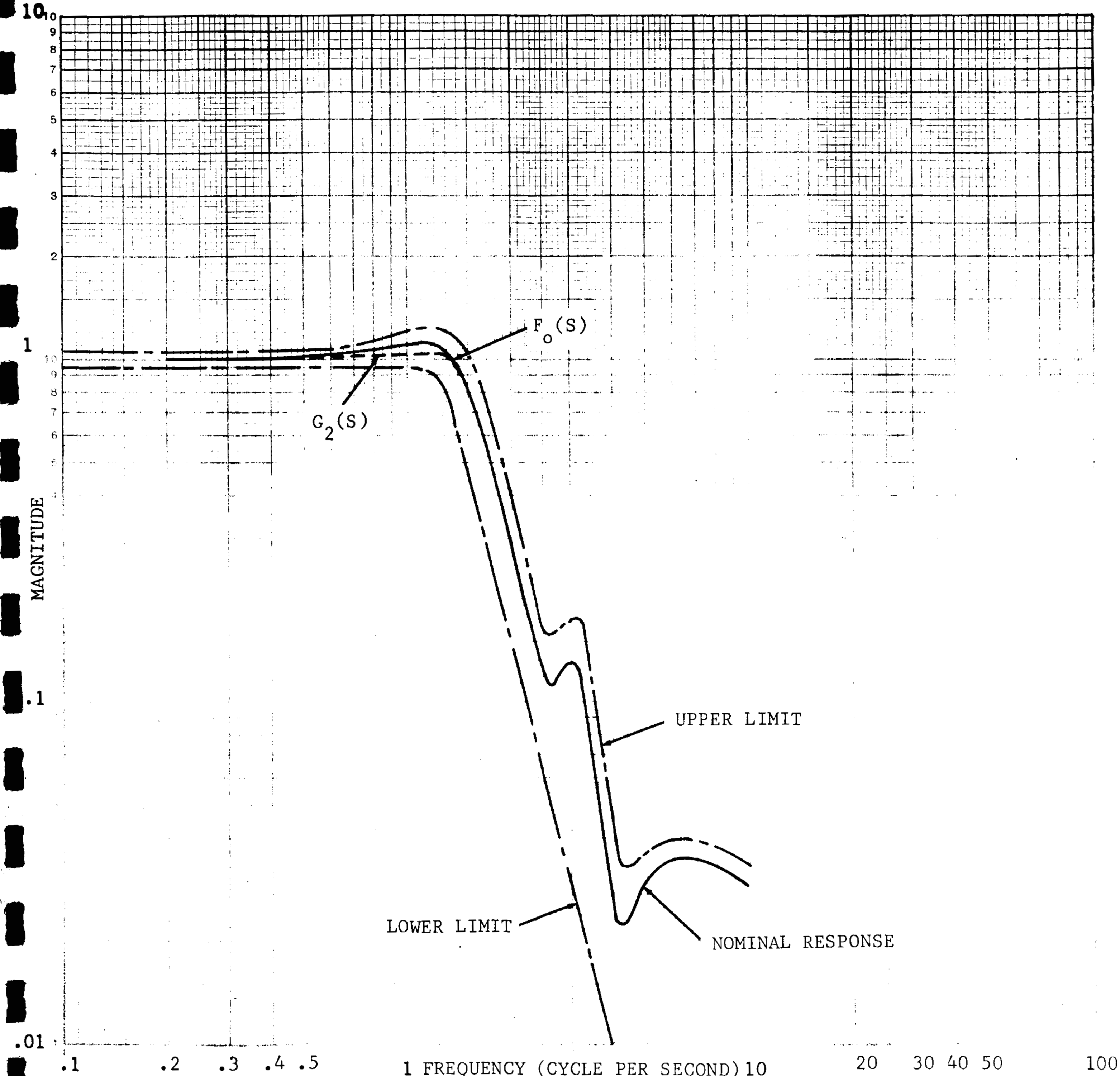



FIGURE 5-14a. SATURN SA-9 PITCH-YAW ATTITUDE RATE SHAPING NETWORK FREQUENCY RESPONSE, S-I STAGE BURN SPECIFICATION LIMITS AND CURVE FIT FOR GAIN

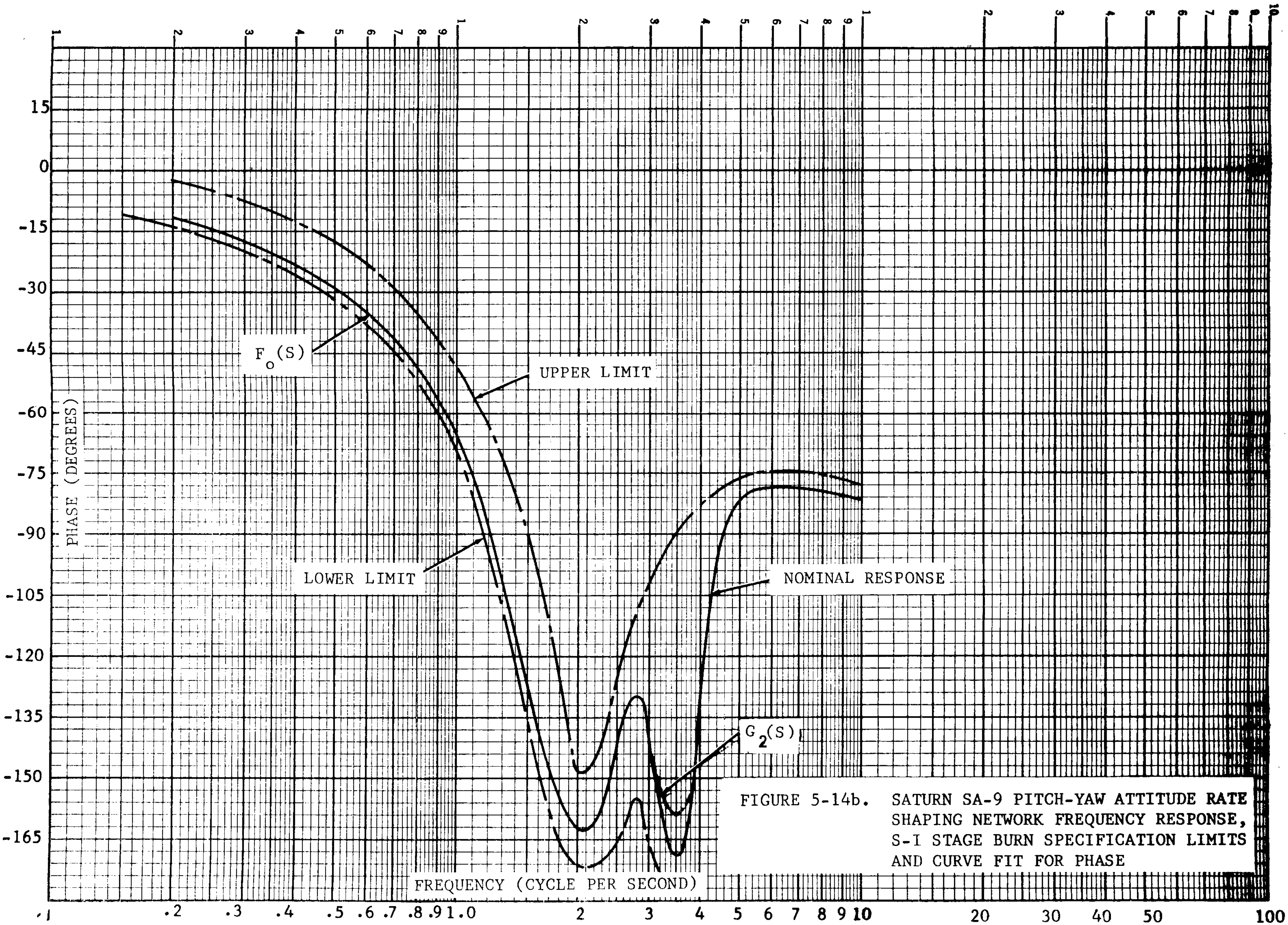



FIGURE 5-14b. SATURN SA-9 PITCH-YAW ATTITUDE RATE SHAPING NETWORK FREQUENCY RESPONSE, S-I STAGE BURN SPECIFICATION LIMITS AND CURVE FIT FOR PHASE

| Zeros | Poles |
| --- | --- |
| $-2.50 \pm 24.87j$ | $-2.54 \pm 19.25j$ |
| $-2.50 \pm 15.61j$ | $-2.87 \pm 8.63j$ |
| | $-10.14 + 0j$ |
| | $-2889.00 + 0j$ |

$$G_2(S) = \frac{0.00000666S^4 + 0.0000725S^3 + 0.00605S^2 + 0.0298S + 1.03}{0.00000298S^5 + 0.0000707S^4 + 0.00182S^3 + 0.0245S^2 + 0.1755S + 1} \tag{5.27}$$

| Zeros | Poles |
| --- | --- |
| $-2.33 \pm 15.45j$ | $-11.96 + 0j$ |
| $-3.11 \pm 25.00j$ | $-2.93 \pm 8.34j$ |
| | $-2.93 \pm 18.71j$ |

The simplest transfer function is (5.27).

### 5.4.2 Condition II, Phase and Gain Specifications Define Discrete Bounded Regions:

The bounded regions, in this case, are defined only at some specific frequency modes. To find a stable transfer function passing through the discrete bounded regions is rather difficult without using templates. The method of the application of templates has been discussed in Section 5.3.1. The following cases of study are accomplished through the application of templates and complex curve-fitting techniques. The results of study are very satisfactory as well.

5.4.2.1 Case 1: Rate Gyro Filter Phase and Gain Specifications. A simplest stable transfer function has been obtained. The detailed results have been discussed as example 2 in Section 5.3.1 and 5.3.2.

5.4.2.2 Case 2: $\alpha$-Meter Filter Phase and Gain Specifications. Through the use of templates a stable transfer function $F_o(S)$ is obtained

$$F_o(S) = \frac{0.0142S^2 + 0.00595S + 1}{0.1600S^2 + 0.02000S + 1} \tag{5.28}$$

Use this transfer function as input to the computer, through complex curve-fitting process, three more stable desired transfer functions are obtained. The results are shown in equations (5.29), (5.30), (5.31), and Figures (5.15), (5.16), and (5.17).

$$G_1(S) = \frac{0.0000435S^3 + 0.0117S^2 + 0.0112S + 0.768}{0.0000157S^4 + 0.000415S^3 + 0.1605S^2 + 0.0356S + 1} \tag{5.29}$$

| Zeros | Poles |
|---|---|
| $-0.36 \pm 8.12j$ | $-0.103 \pm 2.50j$ |
| $-270.42 \pm 0j$ | $-13.11 \pm 100.27j$ |

$$G_2(S) = \frac{0.000047S^3 + 0.01228S^2 + 0.01135S + 0.772}{0.000461S^3 + 0.1604S^2 + 0.03591S + 1} \tag{5.30}$$

| Zeros | Poles |
|---|---|
| $-0.34 \pm 7.93j$ | $-0.10 \pm 2.50j$ |
| $-259.69 \pm 0j$ | $-347.65 + 0j$ |

$$G_3(S) = \frac{0.01227S^2 + 0.00874S + 0.772}{0.16031S^2 + 0.03303S + 1} \qquad (5.31)$$

| Zeros | Poles |
|---|---|
| $-0.36 \pm 7.92j$ | $0.10 \pm 2.50j$ |

One important point should be mentioned here is that, as we can see from Figure (5.15) through (5.17), the nominal gain curve (solid curve) produced by transfer function $F_o(S)$ in equation (5.28) is not well within the bounded regions at the control mode. However, all of the fitted gain curves (dotted curve) are well within specification limits. This again proves the usefulness of complex curve fit process.

5.4.2.3 Case 3: Attitude Gyro Filter Phase and Gain Specifications. Owing to less restrictions caused by the specification limits, only one template ($\zeta = 0.17$, $\omega_n = 5.5$) has been used. The transfer function obtained is shown in equation (5.32)

$$F_c(S) = \frac{1}{0.03306S^2 + 0.06182S + 1} \qquad (5.32)$$

| Zeros | Poles |
|---|---|
| --- | $-0.94 \pm 0.36j$ |

The nominal curve is shown in figure (5.18). Because transfer function (5.32) is simple enough and also is desirable and stable, complex curve fit is not necessary.

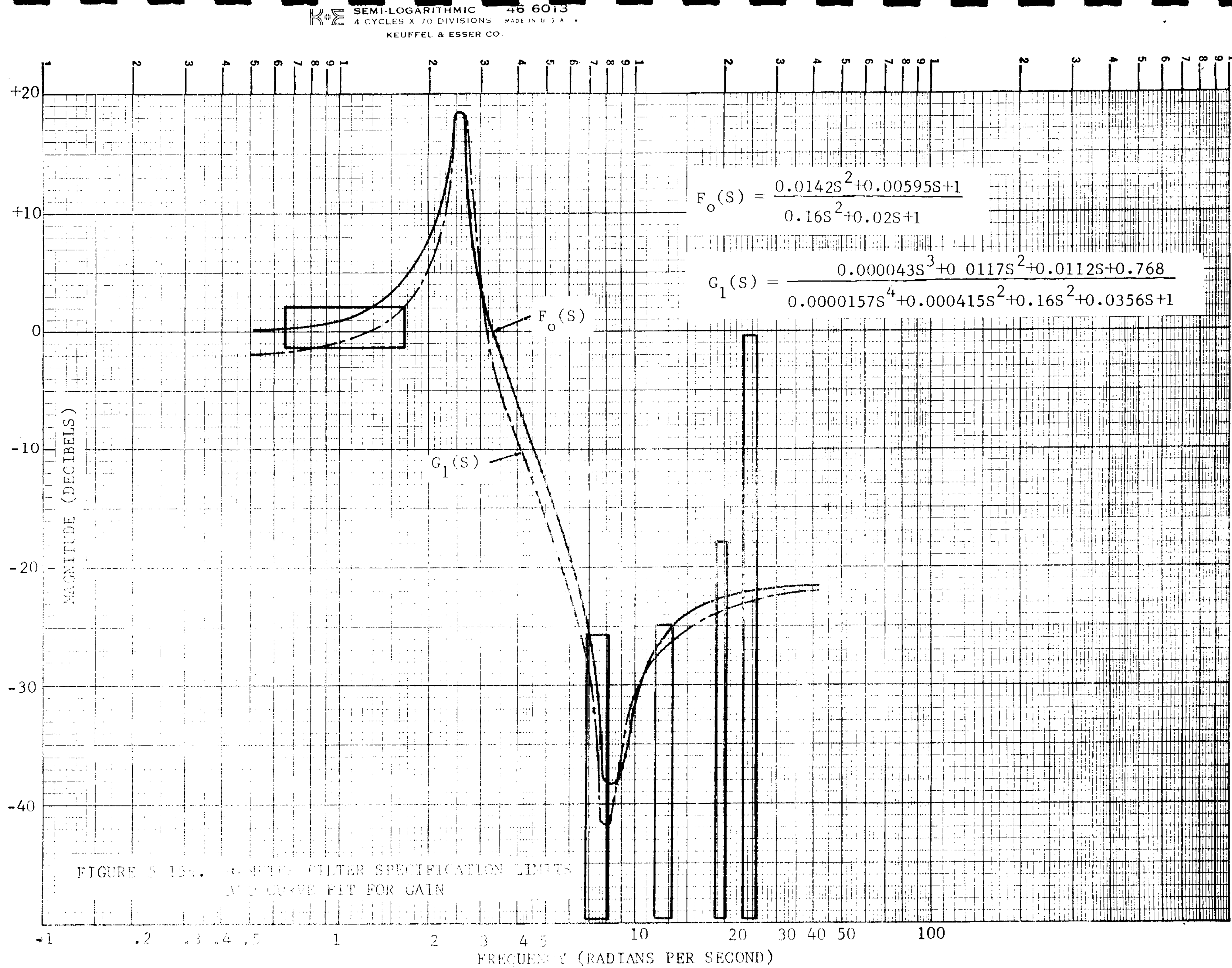



FIGURE 5.15a. [illegible] FILTER SPECIFICATION LIMITS AND CURVE FIT FOR GAIN

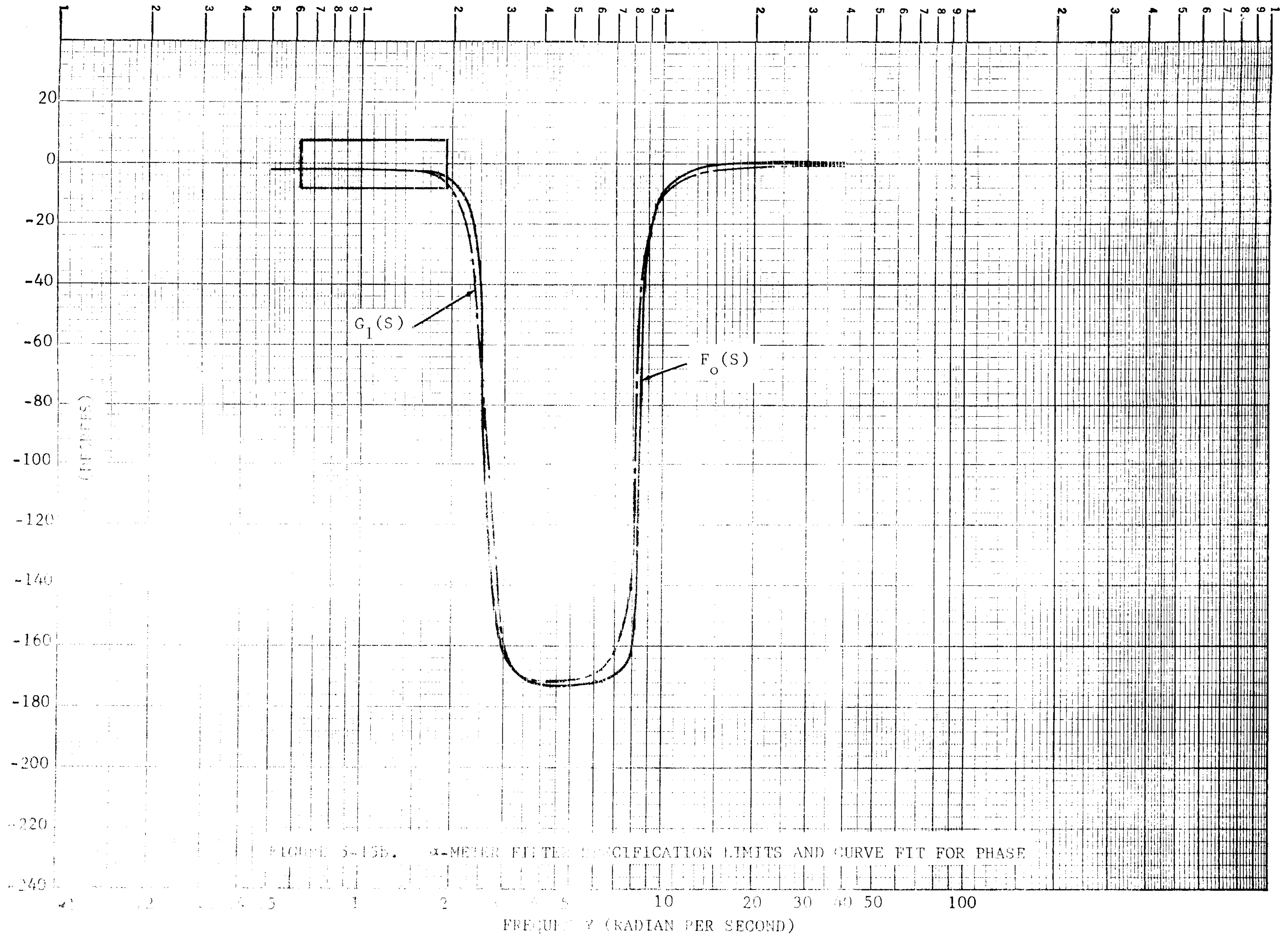



FIGURE 5-15b. α-METER FILTER SPECIFICATION LIMITS AND CURVE FIT FOR PHASE

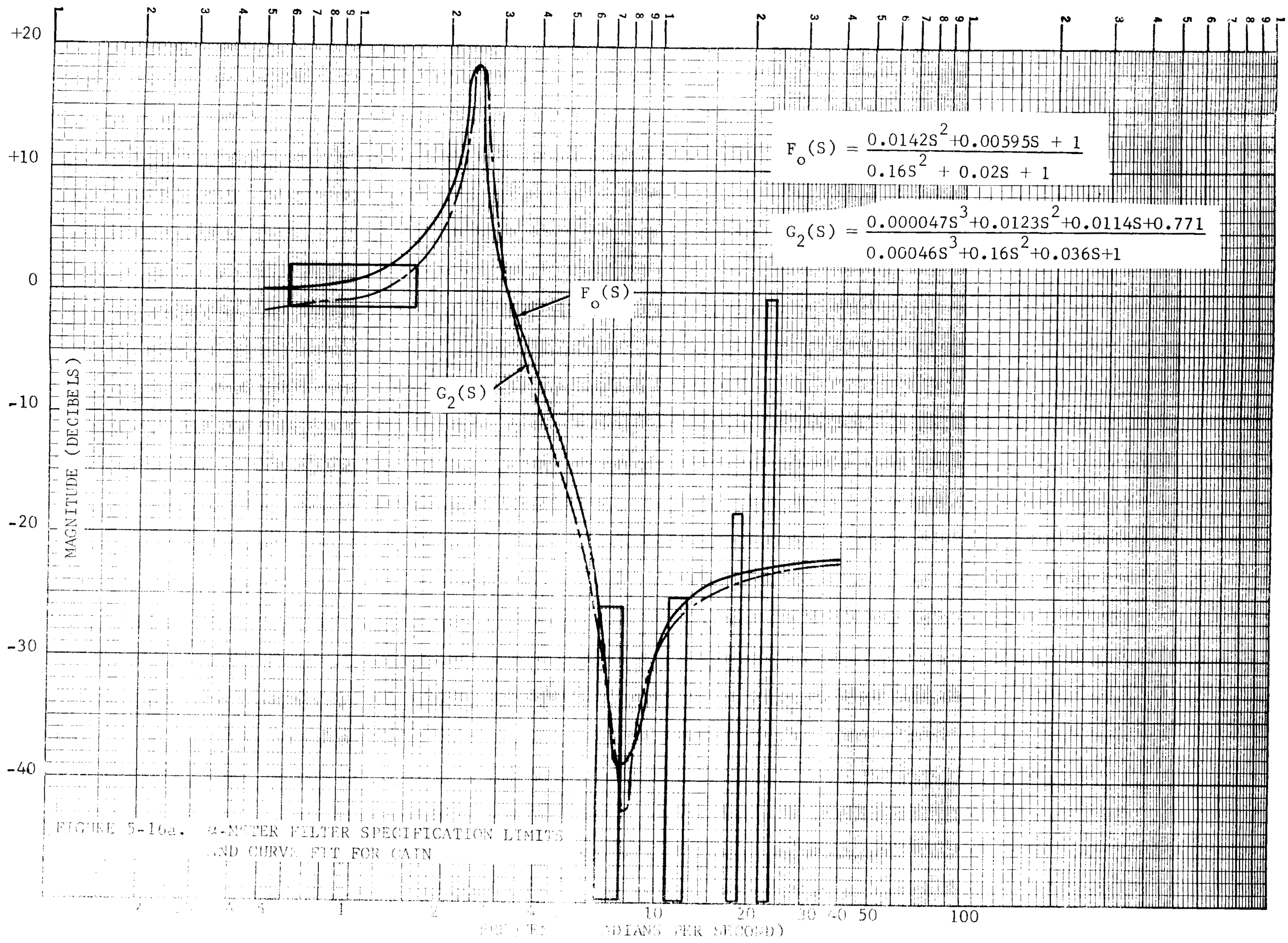



FIGURE 5-16a. 4-METER FILTER SPECIFICATION LIMITS AND CURVE FIT FOR GAIN

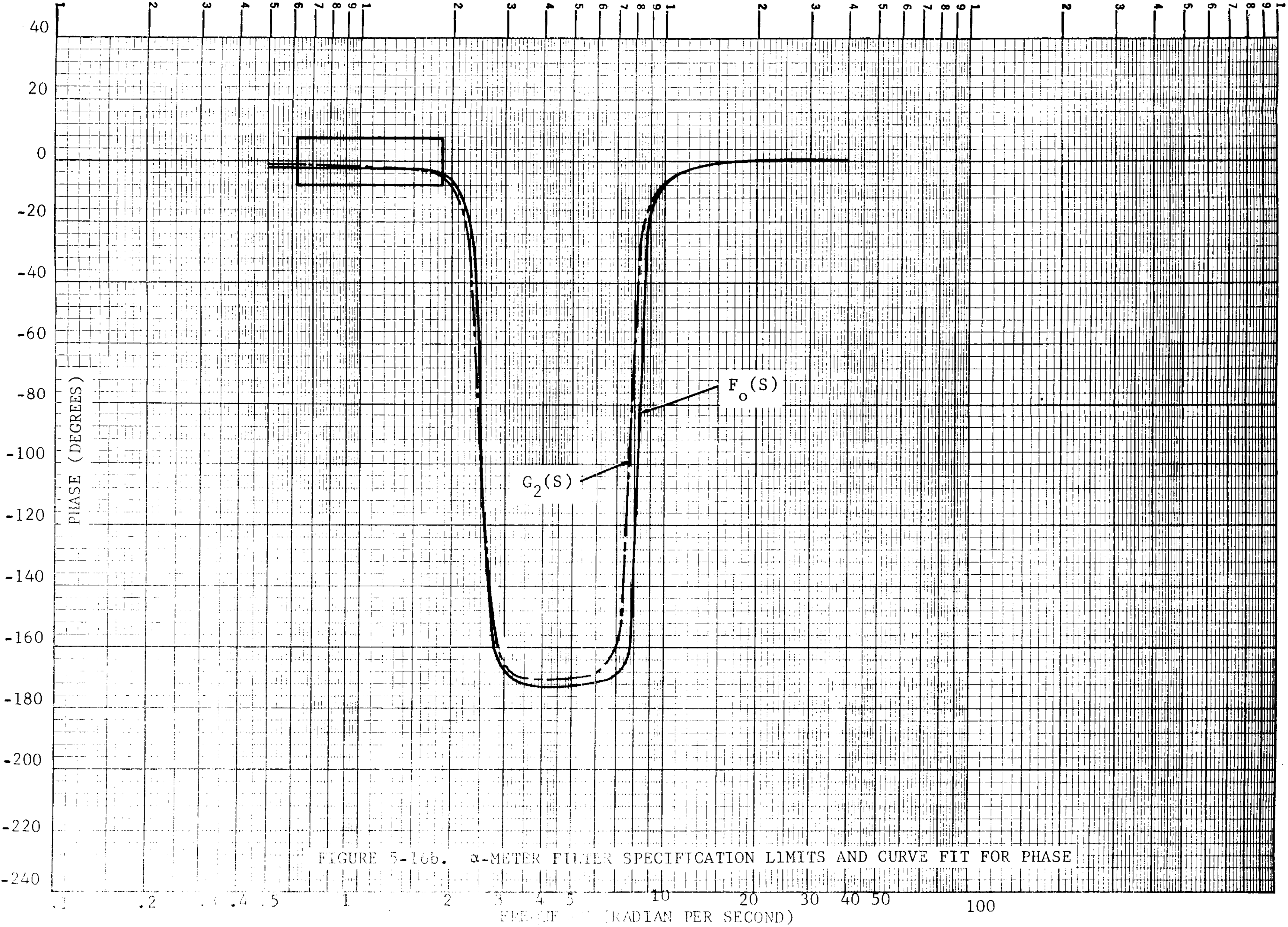
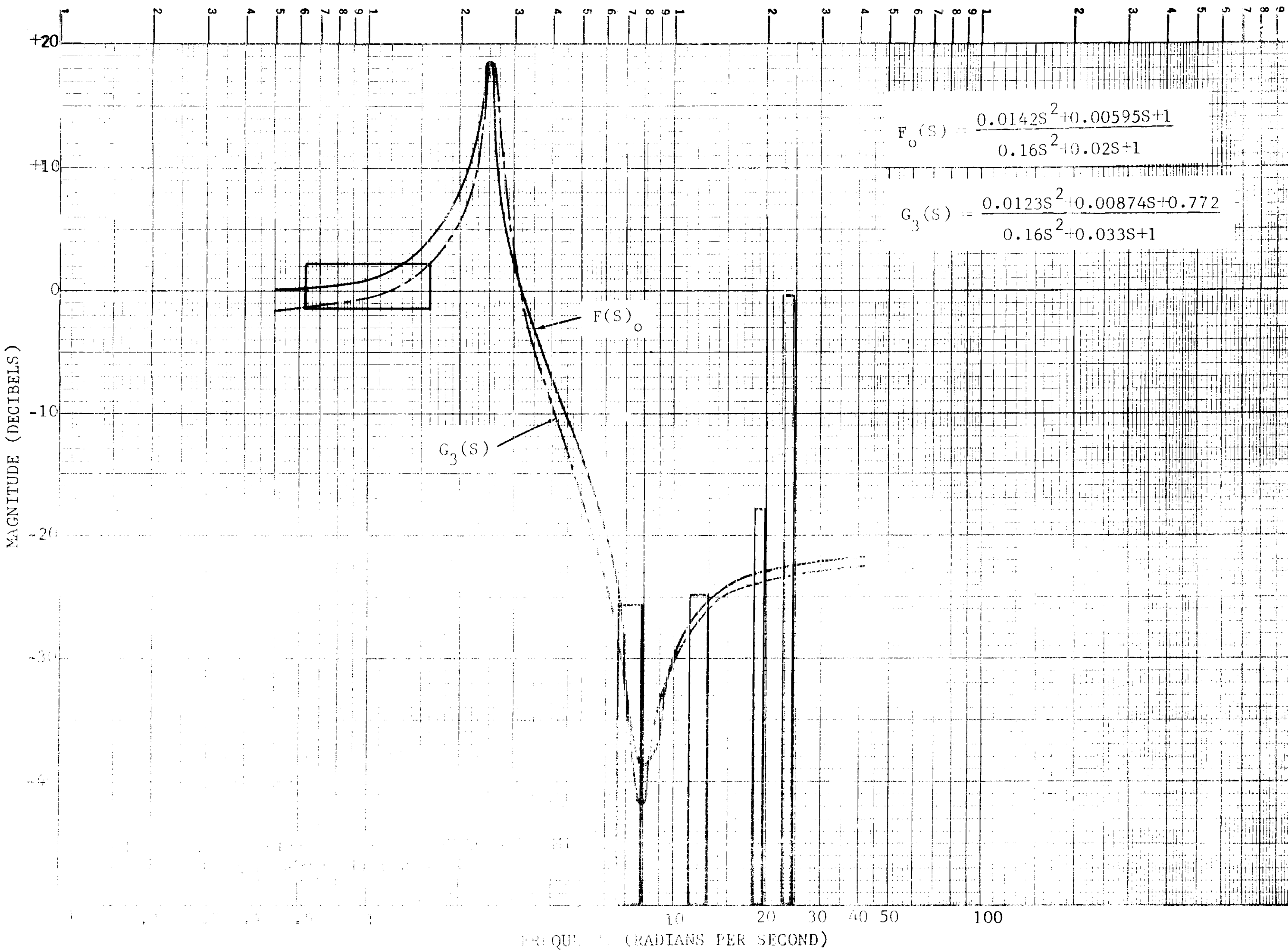



FIGURE 5-16b. α-METER FILTER SPECIFICATION LIMITS AND CURVE FIT FOR PHASE


$F_o(S) = \frac{0.0142S^2+0.00595S+1}{0.16S^2+0.02S+1}$
$G_3(S) = \frac{0.0123S^2+0.00874S+0.772}{0.16S^2+0.033S+1}$
$F(S)_o$
$G_3(S)$
+20
+10
0
-10
MAGNITUDE (DECIBELS)
10
20
30
40
50
100
(RADIANS PER SECOND)

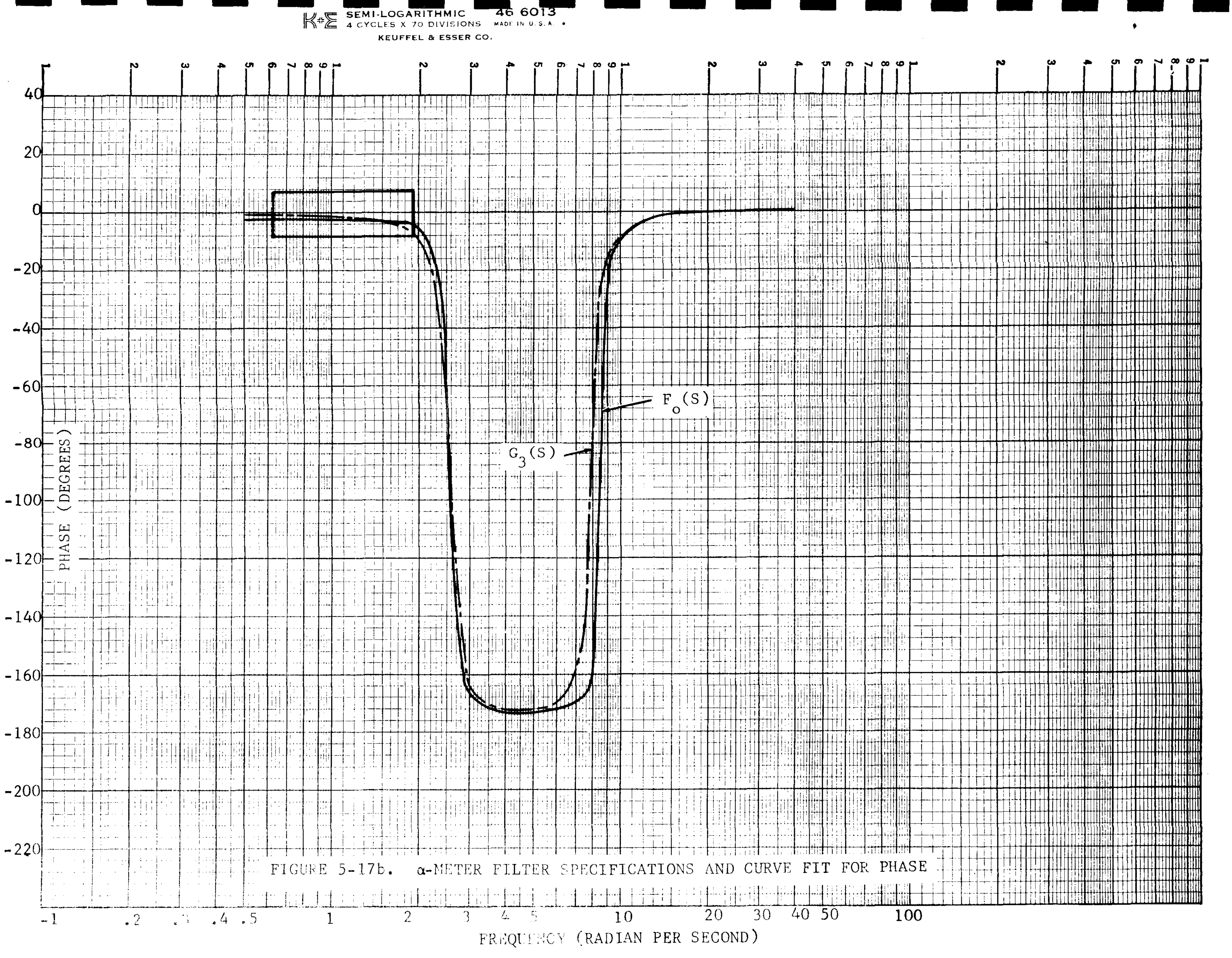



FIGURE 5-17b. α-METER FILTER SPECIFICATIONS AND CURVE FIT FOR PHASE

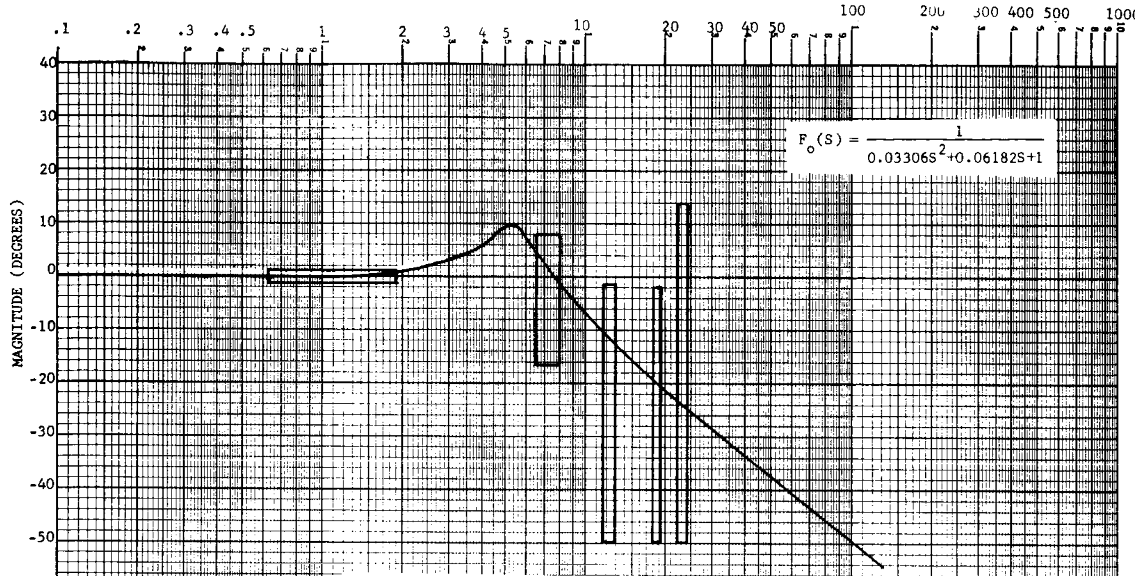



FIGURE 5-18a. ATTITUDE GYRO FILTER SPECIFICATIONS AND CURVE FIT FOR GAIN

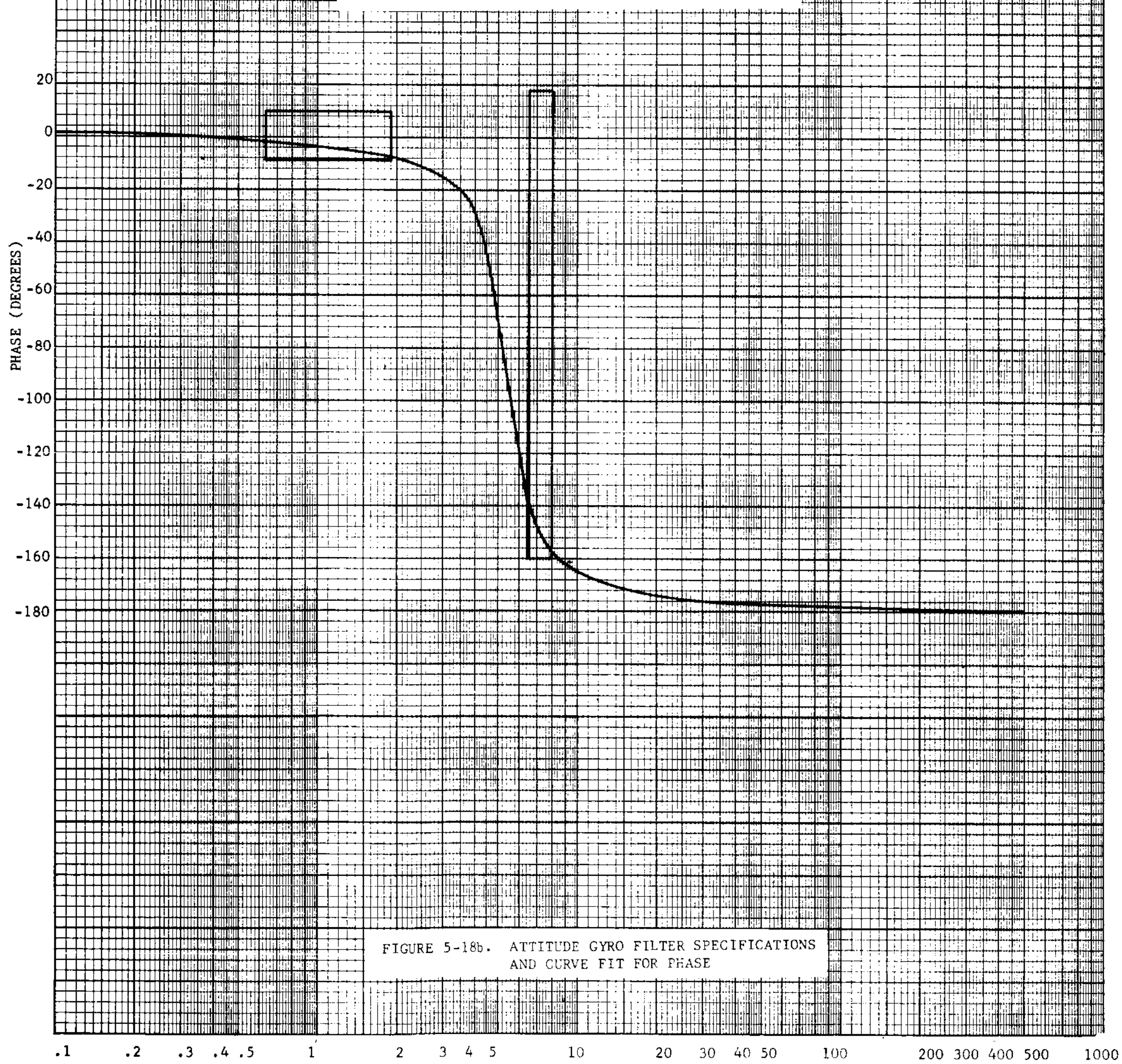



FIGURE 5-18b. ATTITUDE GYRO FILTER SPECIFICATIONS AND CURVE FIT FOR PHASE

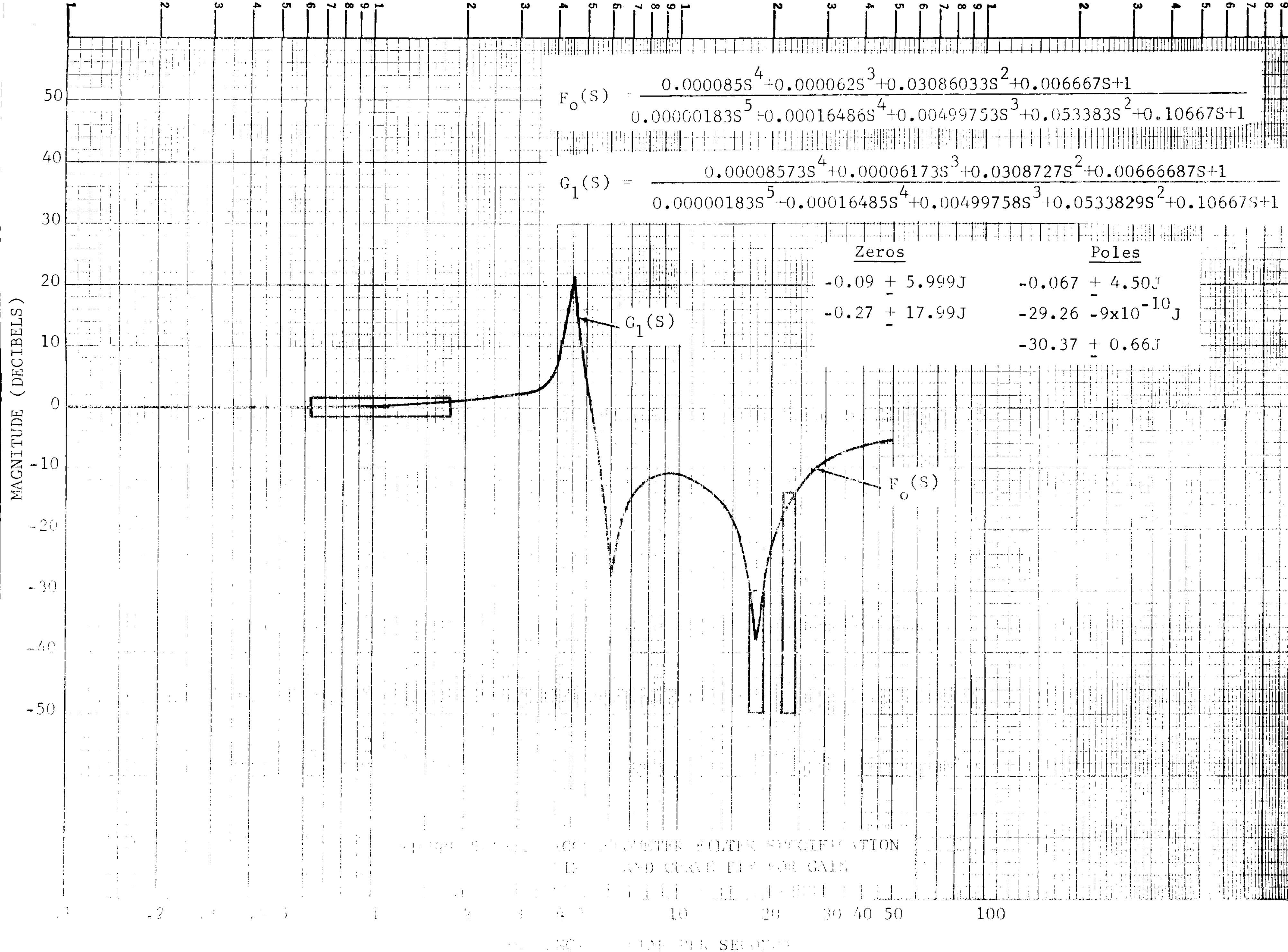
5.4.2.4 Case 4: Accelerometer Filter Phase and Gain Specifications (Modified). Although, in comparison with the other assignments, the specification limits is most restrictive in this case, a stable transfer function has been defined well within the specification limits with slightly more deliberation. The transfer function obtained is shown in equation (5.33)

$$F_o(S) = \frac{0.0000857S^4 + 0.000062S^3 + 0.03086033S^2 + 0.006667S + 1}{0.00000183S^5 + 0.00016486S^4 + 0.00499753S^3 + 0.053383S^2 + 0.10667S + 1} \tag{5.33}$$

Through complex curve fit process, transfer function $G_1(S)$ is obtained which is stable and well within the specification limits.

$$G_1(S) = \frac{0.0000857S^4 + 0.00006173S^3 + 0.0308727S^2 + 0.00666687S + 1}{0.00000183S^5 + 0.00016485S^4 + 0.00499758S^3 + 0.0533829S^2 + 0.10667S + 1}$$

| Zeros | Poles |
|---|---|
| $-0.09 \pm 5.999J$ | $-0.067 \pm 4.50J$ |
| $-0.27 \pm 17.99J$ | $-29.26 - 9 \times 10^{-10}J$ |
| | $-30.37 \pm 0.66J$ |




COMPUTER FILTER SPECIFICATION
AND CURVE FIT FOR GAIN

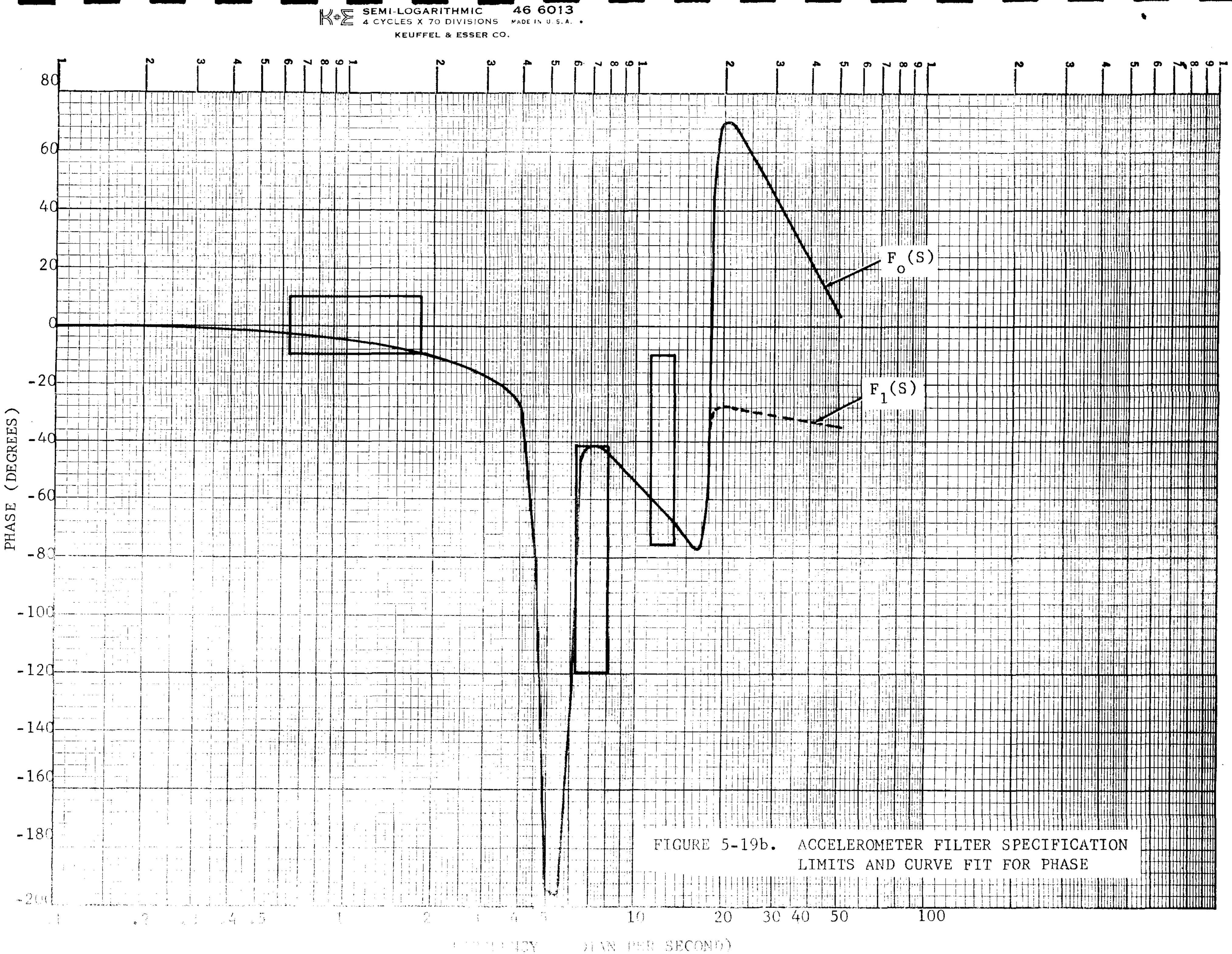



FIGURE 5-19b. ACCELEROMETER FILTER SPECIFICATION LIMITS AND CURVE FIT FOR PHASE

# 6. CONCLUSION

Complex curve-fitting technique is mainly designed for the purpose of extraction of a transfer function from a set of data obtained from experimental results. The data usually includes the values of phase and gain at various frequency points. The transfer function extracted from this data naturally will be stable if the system under experiment is a minimum-phase system. This is the result of the preservation of certain relationships between phase and gain that are necessary for a system to be stable, i.e., the frequency response characterize a minimum-phase system. As a result, the complex curve-fitting technique always has the capability to attain a very close fit to the data. Consequently, the transfer function as obtained will represent the data with great accuracy.

To define a minimum-phase transfer function within the bounded region of phase and gain specifications, the minimum-phase requirements and rules of stability for a transfer function must be observed. To this effect, a set of templates is used. Through complex curve fit process a simplest and more refined desirable transfer function can be obtained within the phase and gain specification limits.

The only weak point of this method is that without the application of templates one can hardly define a stable transfer function within a discrete region of a given phase and gain specification. The use of templates needs some manual manipulation; however, it is believed that with additional study, analytical expressions which represent the templates may be incorporated into the computer program for complex curve fitting and the technique for defining a stable transfer would be fully automated.

## 7. REFERENCES


1. Wilcox, J., Tung, S., and White, K., "Network Synthesis and a Method to Define a Transfer Function from Prescribed Phase-Magnitude Characteristics", NSL/Huntsville Aero/Astro Tech. Memo. No. 16, May 1964.

2. Hosenthien, Hans H., "Control Shaping Network Change for Saturn S-9 S-I Stage", Memo # R-ASTR-F-50, NASA, Huntsville, July 1964.

3. Hosenthien, Hans H., "SA-9 S-I Stage Control Gains and Shaping Networks with Associated Pole-Zeros, Frequency Responses, and Response Tolerences", Memo. # R-ASTR-F-46, NASA,Huntsville, July 1964.

4. Yeh, Victor C. M., "The Study of Transients in Linear Feedback System by Conformal Mapping and the Root-Locus Method", Transaction of the American Society of Mechanical Engineering, April 1954.

5. D'Azzo, John J., and Houpis, Constantine H., Feedback Control Systems Analysis and Synthesis, McGraw-Hill Book Co., Inc., New York, 1960.

6. Truxal, John G., Editor-in-Chief, Control Engineers' Handbook, McGraw-Hill Book Co., Inc., New York, 1958.

7. Ryan, Roberts S., "Numerical Procedures for Stability Studies", Technical Memorandum X-53109, George C. Marshall Space Flight Center, Huntsville, August 1964.

8. Ryan, Robert S., "Stability Considerations of a Space Vehicle in Bending Oscillation for Various Control Sensors", Technical Memorandum MTP-AERO-62-64, George C. Marshall Space Flight Center, Huntsville, Alabama, August 1962.

9. Hayes, Phillip J., "Stability Analysis of Saturn SA-6 with Rate Gyro for S-IV Control Damping", Technical Memorandum X-53054, George C. Marshall Space Flight Center, Huntsville, Alabama, June 1964.

10. Sanathan, C. K., and J. Hoener, "Transfer Function Synthesis as a Ratio of Two Complex Polynomials", IEEE Transactions on Automatic Controls, January 1963.